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UNIVERSITY OF ALBERTA

SOLUTIONS OF THE 3-DIMENSIONAL TIME-DEPENDENT LANDAU-GINZBURG EQUATION FOR REAL ORDER PARAMETERS OBTAINED BY SYMMETRY REDUCTION

BY

MACIEJ SKIERSKI ©

A THESIS
SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH
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OF

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DEPARTMENT OF PHYSICS

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FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommended to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled "Solutions of 3-dimensional time-dependent Landau-Ginzburg equation for real order parameters obtained by symmetry reduction" submitted by Maciej Skierski in partial fulfilment of the requirements for the degree of Doctor of Philosophy in Theoretical Physics.

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ABSTRACT

This work contains the results of the symmetry reduction of the time- and space-dependent 3 + 1 dimensional real Landau-Ginzburg equations with positive and mixed signatures. These results consist of a list of representatives of the conjugate classes of subalgebras of the corresponding Lie algebra. All subalgebras leading to reductions to algebraic or first and second order differential equations are identified and corresponding reductions are performed explicitly. Geometries of the surfaces of constant value of the symmetry variables are described in some cases. Solutions to most of the first order differential equations are written down. The method of solution of the remaining first order equations is indicated. The Painlevé test is applied to all second order differential equations. Equations for which the result of this test was positive were solved. Numerical attempts to estimate the behaviour of the free energy for solutions with three different geometries for each signature are also reported. The results of these calculations were compared with each other to determine the influence of the signature on the nature of the states.

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CHAPTER ONE

INTRODUCTION

The main goal of this thesis is to present solutions of the time and space-dependent three dimensional Landau-Ginzburg equation obtained by application of the symmetry reduction method.

In this work the time- and space-dependent Landau-Ginzburg equation appears as an equation of motion of a non-conserved order parameter.

We are interested in the Landau-Ginzburg equation as a very simple model of continuum description of phase transitions using a coarse-grained approach. There is an intensive effort of the group led by J.A. Tuszyński to investigate mathematical and physical properties of this nonlinear model. His works published together with A.M. Grundland, P.Winternitz [66] and also the MSc thesis of J.Rendell [56] deal with the symmetry reduction analysis of this equation. This thesis is an extension of this direction of research. The results of the application of symmetry analysis to the time-dependent 1+3 dimensional Landau-Ginzburg equation are presented in the second chapter and are the main part of this thesis.

On the other hand the analysis of the energy values, the partition function associated with specific (Jacobi type) solutions was actually performed by A. Nip and J.A. Tuszyński [46]. Stability properties of these solutions were investigated by D. Schwartz and J.A. Tuszyński [58]. The preliminary comparison of some results of the energy calculations is presented in the last chapter of this thesis.

Recent developments in the area of symbolic computation allowed wider applications of the symmetry reduction method. A large part of these calculations can be performed by a computer. Since this method became computationally feasible many equations of mathematical physics have been analyzed by means of it. For example the self-dual Einstein equations were analyzed by C. Boyer et al in [9], the Wess-Zumino model by M. Légaré in [42], the nonlinear Schrodinger equations with a cubic or quintic nonlinearity by L. Gagnon et al [16,17,18].

Out of these investigations the closest to the present one are works in Ref. [16,17,18,66,67] In the series of papers by L.Gagnon et al [16,17,18] solutions to the nonlinear Schrodinger equations were discussed using the symmetry reduction method. The functional form of the nonlinear Schrodinger equation is very similar to the Landau-Ginzburg equation. The NLS deals, however, with complex variables while the Landau-Ginzburg equation of interest to us is purely real.

It should be mentioned that symmetry reduction is only one of the methods that can be applied to obtain analytic solutions of non-linear partial differential equations. It is possible to apply other methods such as: Lie-Bäcklund transformations [5] or new similarity reductions [12].

The plan of this thesis is as follows.
In this introductory chapter main ideas, such as phase transition, order parameter, and coarse grained quantities, will be introduced. The limitations and extensions of their applications will be discussed as well. The discussion of the relation between phase transitions and symmetry groups follows. Then some side issues related to phase transitions and to the Landau theory such as the Landau-Ginzburg criterion, the continuous Ising model and critical indices are briefly described. The next section contains discussion of the equation of motion.

The second chapter contains a very general description of the symmetry reduction method. It begins with a simple example illustrating the power of the symmetry reduction method applied to the 1+1 dimensional linear heat equation. The main steps of the symmetry reduction method will be introduced in this case. The more precise formulation is deferred to the appendix. The following section contains the description of the relation between the Lie subalgebras and reductions of the differential equation. The discussion of analysis of the algebraic structure of the Lie algebra follows. This chapter closes with a brief description of the Painlevé test.

The following chapter contains the main results of my thesis. The first part describes the results of the symmetry analysis applied to the Landau-Ginzburg equation with positive signature of the gradient term. This part starts from a discussion of generators of the relevant Lie algebra. The structure of the system of its subalgebras follows. Then reduced ordinary equations obtained using subalgebras of the proper dimension are listed. Those reduced ordinary differential equations that satisfy the Painlevé test are then solved.

The second part of this chapter contains results of the symmetry analysis of a modified Landau-Ginzburg equation with mixed signature of the gradient term. Discussion of its properties follows the same path as the similar discussion in the first part.

The fourth chapter contains a comparison of the numerical values of the free energy functional for a few chosen reductions for both signatures. The possibility of transitions between these two cases is also discussed in this chapter.

The conclusion which contains a short discussion of implications of the obtained results closes the body of text.

The appendix consists of two parts. The first part contains some basic mathematical definitions which are required in the following theorems. In the second part I list some main theorems showing relations between the differential equation and its Lie algebra.

1.1 Time and space dependent phenomena in phase transitions

The following two subsections contain descriptions of the fundamental approaches used in the analysis of large systems on different length and time scales.

a) Thermodynamics

The behaviour of large systems is usually described using one of two approaches. The first one, the thermodynamical approach, is applicable to systems that are close to equilibrium and evolve on a large time scale in comparison to local equilibration time (quasi-static approximation).

The thermodynamic approach was initially constructed to describe highly homogeneous systems. It assumed that the state of the whole system can be described using a very small number of extensive and intensive variables. Obviously, to each thermodynamic state there correspond an enormous number of microscopic states.

In thermodynamics the main role is played by two types of objects. The first one is the state function of a system. The second type are the so-called thermodynamic potentials. Both objects are functions of thermodynamic variables. The knowledge of the state function and one of the thermodynamic potentials is sufficient for calculations of the physically measurable susceptibilities. Susceptibilities are expressed in terms of derivatives of different thermodynamic potentials. Using the Lagrange transformation one can transform each of the potentials into any other.

Thermodynamics is, however, a phenomenological theory in the sense that thermodynamic potentials and the state function are taken from the experimental results. One can find justification of the results obtained from classical thermodynamics deriving laws of thermodynamics from more fundamental laws of statistical physics. However, practical calculations of the thermodynamic potentials from the microscopic principles are, with the exception of a few systems (e.g. ideal or nearly ideal gases), very difficult and one has to use the thermodynamical methods.

The ergodic property is one of the most essential assumptions of statistical mechanics. It means that it does not matter if one calculates an average value of any physical quantity over many identical systems (replicas) or if one uses only one system but averages over long times of evolution. It is believed that most equilibrium systems possess the ergodic property.

The main limitation of the thermodynamical approach which is the homogeneity of a state of a system on the length scale comparable with the size can be relaxed if one uses the coarse grained approach.

b) Coarse grained approach

Introduction of the coarse grained quantities allows discussion of spatially-modulated phenomena. The spatially homogeneous thermodynamic variables are dropped and new variables which are slowly varying on the length scale L are used instead. Such an approach is justified when there exists a finite range interaction that takes blocks of the dimension L^d back to equilibrium in a short time compared to the time scale of changes taking place over long distances.

The next few paragraphs contain an interpretation of the main equations of this thesis as it was given by Gunton and Droz in their book [23]. Both derivation of those equations and their physical interpretation are based on the concept of the coarse-grained free energy.

The partition function of the system is given as

$$Z = \sum_{\{m_i\}} \exp(-\beta H\{m_i\}) \tag{1.1}$$

where summation is over the all possible configurations of the variable m_i and $\beta = (kT)^{-1}$. In the above equation i numbers the lattice sites. The free energy F of such a system is given by the equation $F = -\beta^{-1} \ln Z$. The free energy function is therefore a function of intensive parameters such as $\beta = (kT)^{-1}$. It does not depend on the actual value of the distribution m_i. It is clear that one cannot discuss the spatially varying distributions using the free energy function defined in this way.

Another way of defining a physically interesting functional is based on local averaging of the distribution m_i. One starts by dividing the sample into a large number of subregions that are large in comparison with the lattice constant and small in comparison both with the size of the sample and the correlation length. In these subregions one defines the average values m_α of the distribution m_i. After summation over the microscopic configurations m_i compatible with a constraint m_α the partition function given in Eq. (1.1) can be rewritten in the form:

$$\begin{aligned} Z &= Tr_{\{m_\alpha\}} W\{m_\alpha\} \exp(-\beta E\{m_\alpha\}) \\ &= Tr_{\{m_\alpha\}} \exp[-\beta F_\alpha\{m_\alpha\}] \end{aligned} \tag{1.2}$$

where the symbol $W\{m_\alpha\}$ denotes the number of configurations compatible with a local average value m_α and local functionals F_α are defined as

$$F_\alpha(m_\alpha) = -\beta^{-1} \ln \sum_{\{m_i\}}{}' \exp(-\beta H\{m_i\}).$$

In the last equation $\sum'$ denotes the summation over the microscopic configurations compatible with the constraint m_α. One can then define the global free energy as the functional $F(m) = \sum_r F_r(m_r)$. This functional is defined in the space of average distributions which in turn are defined over the whole sample. The functional F_α defined in Eq. (1.2) is called *the coarse-grained free energy*. One should be aware that this object is quite different from the free energy function defined in thermodynamics. It is distinguished by its dependence of the on the locally averaged values of the distribution. However, if one considers only spatially homogeneous solutions then the local and global averages are equal and the thermodynamic and coarse-grained free energies differs only by an L dependent constant.

Transition to the continuous description takes place at the level of the averaged local densities. Therefore one obtains the coarse-grained free energy functional defined over continuous distributions m_α.

One, however, rarely calculates the coarse-grained free energy starting from a microscopic model. It is generally accepted to assume the Landau-Ginzburg form of this functional as

$$\begin{aligned} F_L\{m(\vec{r})\} &= \int f_L\{m(\vec{r})\} \\ &= \int d^d x[\frac{1}{2} c_L |\nabla\{m(\vec{r})\}|^2 + V_L\{m(\vec{r})\}. \end{aligned} \tag{1.3}$$

The index L expresses the fact that the coarse-grained free energy depends on the size of the cell L used to average.

According to M. Luban [43] the coarse grained order parameter can be defined using the following approximation

$$m(\vec{r}) = \sum_{|\vec{k}|<k_c} \exp(-i\vec{k}\cdot\vec{r}) m_{\vec{k}}$$

where k_c is a cutoff value for the vector $\vec{k}$ and where $m_{\vec{k}}$ is a Fourier transform of the discrete system of values $m(\vec{r})$ indexed by $\vec{R}$

$$m_{\vec{k}} = N^{-1} \sum_{\vec{R}} \exp(i\vec{k}\cdot\vec{r}) m_{\vec{R}}.$$

A more detailed definition of the order parameter is given in the next section but at this moment it is convenient to think about it as just another thermodynamic variable even if there are some significant differences between these two notions.

The main difference here in the approach of J.D. Gunton and M. Droz [23] or that of J.S. Langer [41] is that the coarse graining operation is performed in the Fourier conjugated space of the variable $\vec{k}$.

If one denotes the equilibrium free energy as F and the coarse grained free energy as F_L, then the partition function can be expressed as

$$Z = \exp -(\beta F) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{|\vec{k}|<k_c} dm_{\vec{k}} \exp\left[-\int_{L^d} d^d\vec{r} F_L\right] . \tag{1.4}$$

c) Singularities of the thermodynamic potentials. Phase transitions.

It may happen that derivatives of a thermodynamic potential have singularities. The points at which singularities appear are called ***phase transition points.*** At phase transition points some physically measurable properties quantities change their values in a discontinuous fashion. Traditionally phase transitions were classified with respect to the order of derivative of the free energy at which discontinuity appears.

A *phase transition* is a process in which some physical properties of a system change in a discontinuous way. In a traditional (Ehrenfest) classification of phase transitions different classes of transitions are differentiated by the lowest order derivatives of the thermodynamic potential which are discontinuous at the phase transition point. And so, for first-order phase transitions some of the first order derivatives (e.g. entropy, energy, or volume) of the thermodynamic potentials are discontinuous. Second-order phase transitions are related to continuous changes of the previously mentioned thermodynamic variables but discontinuity of the second derivatives of the thermodynamic potentials occurs. There are many different theories describing phase transitions. For example the best description of the critical exponents is presently given by methods of the renormalization group. On the other hand, a qualitatively good description of the ground state for different system can be achieved by the means of a much simpler mean field theory. Landau's theory belongs to the latter type.

d) Landau theory of phase transitions

In the 1930's L.D.Landau [38] suggested that in order to distinguish between different phases one should append one more variable, called an *order parameter* to the arguments of a thermodynamic potential. However, the order parameter is not a thermodynamic variable because the equation of state does not depend on it. The early theory of phase transitions assumed that it is a quantity characteristic of the whole system in the same way as temperature or pressure is defined for the entire system and is assumed to be equal in all its parts.

Introduction of the order parameter removes singularities from the potential itself. The homogeneous phases are characterized by values of the order parameter that minimize the thermodynamic potential for fixed value of other variables. This section contains a brief description of the Landau theory in its original version. It is based on L.D. Landau and E. Lifschitz's book [40] and on the original paper by L.D. Landau [38].

The concept of order parameter η is of primary importance in this theory. Initially [38] the order parameter was as an additional variable which characterized the degree of order in a system. It was supposed to be zero when the system was in the state of the complete disorder and positive otherwise. Later, Landau [36] extended the range of the variable to the whole real axis. In this later paper the order parameter was given a physical interpretation. Its meaning was illustrated in Landau's book in the following way:

> In transitions where the atoms are displaced from their positions in the symmetrical phase, η may be taken as the amount of displacement. For

transitions with a change in the ordering of the crystal (e.g. in the $CuZn$ alloy), the parameter η may be defined as $\eta = (w_{Cu} - w_{Zn})/(w_{Cu} + w_{Zn})$ where w_{Cu} and w_{Zn} are the probabilities of finding a copper atom and a zinc atom, respectively, at any given lattice site. For magnetic transitions, η may be taken as the macroscopic magnetic moment per unit volume of a ferromagnet or the magnetic moment of the sub-lattice for an antiferromagnet.

From the definition of the order parameter, the value at which the thermodynamic potential has its minimum determines a stable phase. In this way it is possible to distinguish between different phases by calculating the value of the order parameter.

In Landau's theory one investigates a thermodynamic potential $\Phi(P, T, \eta)$. The minimum of this potential defines the value of the order parameter η for each value of P and T. That means that the conditions for equilibrium are given by

$$\frac{\partial \Phi}{\partial \eta} = 0 \qquad \frac{\partial^2 \Phi}{\partial \eta^2} > 0.$$

It is assumed that close to the point of a second-order phase transition this potential can be expanded into a power series with the respect to η.

$$\Phi(P, T, \eta) = \Phi_0 + \alpha\eta + A\eta^2 + C\eta^3 + B\eta^4 + \cdots$$

where the coefficients $\alpha, A, B, C, \ldots$ are functions of P and T. One should not expect this expansion to be always correct or possible but in many cases it yields correct results (e.g. structural phase transitions).

For many applications of the Landau theory using the average (equilibrium) values of thermodynamic variables is quite sufficient. The following example is discussed in L.D. Landau's book [40]: the expansion of Φ contains only terms up to the fourth order in η.

$$\Phi(P, T, \eta) = \Phi_0 + \alpha\eta + A\eta^2 + C\eta^3 + B\eta^4 \qquad (1.5)$$

It can be shown that, if the states corresponding to $\eta = 0$ and $\eta \neq 0$ minimizing the thermodynamic potential are of different symmetry and the transition is continuous, the first-order term α in the above expansion must be identically zero. The coefficient $A(P,T)$ in the second-order term must vanish at the transition point since in the low-temperature phase the value $\eta = 0$ must correspond to a minimum of Φ, and it is necessary that $A > 0$, while on the other side of the transition point, in the high-temperature phase, non-zero values of η must correspond to to the stable state, and this is possible only if $A < 0$. Figure 1.1 contains a sketch of the generic behaviour of the thermodynamic potential Φ close to a second order phase transition. The coordinate axes are taken to be the temperature T and η the value of the order parameter.

However, if the transition point is a stable state, it is necessary that the third order term should be zero and the fourth-order term be positive there:

$$A_c(P,T) = 0, \qquad C_c(P,T) = 0, \qquad B_c(P,T) > 0.$$

There are two possible cases now. First, the third-order term is identically zero owing to the symmetry of the body: $C(P,T) = 0$. Then, at the transition point only the condition $A(P,T) = 0$ remains, which determines P as a function of T or vice versa. Thus, in the PT-plane there is a line of phase transitions of second order.

If, however, C is not identically 0, the transition points are determined by the two equations $A(P,T) = 0, C(P,T) = 0$. In this case, therefore, the continuous phase transition can occur only at isolated points.

For magnetic systems the thermodynamic potential of the body cannot be altered by time inversion. But, the magnetic moment, being related to the electric current, changes its sign under such an operation. For this reason the expansion of Φ contains no odd-terms.

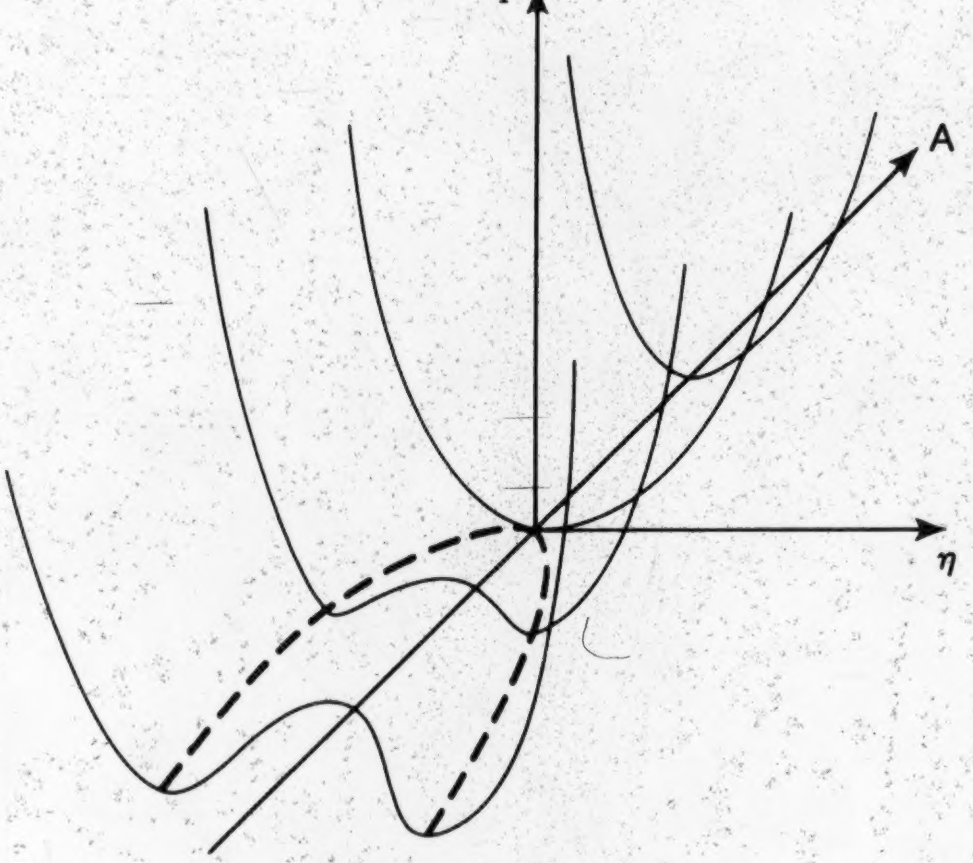

Figure 1.1: Schematic pictures of second order phase transitions in Landau theory plotted in the A, η, Φ space

In the theory given here, it is assumed that the function $A(P,T)$ has no singularities at the transition point, so that it can be expanded near this point in integral powers of temperature

$$A(P,T) = a(P)(T - T_c)$$

where $T_c = T_c(P)$ is the transition temperature. The above temperature dependence of A was proposed by L.D. Landau as the simplest possibility which yields a sign inversion of A. At the same time it was assumed that the coefficient $B(P,T)$ is only weakly temperature dependent in the vicinity of the critical point and it may be replaced by $B(P,T_c)$. The expansion of the thermodynamic potential therefore becomes

$$\Phi(P,T,\eta) = \Phi_0(P,T) + a(P)(T - T_c)\eta^2 + B(P)\eta^4 \tag{1.6}$$

with $B > 0$.

The dependence of η on temperature near the transition point, in the low-temperature phase, is determined from the condition for Φ to be minimum as a function of η. Hence, we obtain $\eta^2 = -A/2B = a(T_c - T)/2B$, while in the high-temperature phase $\eta = 0$ as was required in the first place.

It is also possible to include in this picture first order phase transitions. One can do this by assuming that the coefficient C standing by the third order term in Eq. (1.5) is temperature dependent instead of the coefficient A. This would result in an additional free energy minimum appearing when $A > 0$ and B is sufficiently large and negative. This behaviour was not discussed by L.D. Landau as it yields a first order phase transition and because it involves a non-trivial third order term which is usually forbidden due to symmetry properties (e.g. time inversion). A sketch of the generic behaviour of the thermodynamic potential Φ is given in Fig. 1.2. One can see there the existence of two characteristic values of C. At the first one C_* there is an inflection point. For values $C < C_*$, Φ has two minima. The second characteristic point C_c denotes the point at which the global minimum is at the value a $\eta \neq 0$. In the region between C_* and C_c, $\eta = 0$ is still the global minimum and represents the stable state.

At this point I would like to summarize the main points of the Landau theory of phase transitions,

1. Existence of the order parameter η
    - in the high-temperature (usually high-symmetry) phase $\eta = 0$
    - in the low-temperature (usually low-symmetry) phase $\eta \neq 0$
2. The value of η is given by the minimum of the thermodynamic potential Φ which can be expanded in a series around the point of the phase transition
3. Thus expansion coefficients must satisfy
    - $A(P,T) = 0$ at $T = T_c$
    - $C(P,T) = 0$ if we have a line of second order phase transitions
    - $C(P,T) \neq 0$ if we have isolated point of continuous phase transition
    - $B(P,T) > 0$ at $T = T_c$.

Much later, in 1972 R. Thom [61] gave fundamentals to the classification of singularities of functions defined on a space which is less then 7 dimensional. These results are known as "Catastrophe Theory" [19]. According to this theory, under some assumptions, one can write a generic (up to diffeomorphism) form of function which both the first derivative and the jacobian are zero. One of the simplest elementary catastrophes has the exact form of the Landau 4^{th} degree expansion. Another catastrophe known as ***the butterfly catastrophe*** corresponds to the 6^{th} order expansion. Thus, the previously used term of the generic behaviour of the thermodynamic potential Φ close to phase transition may be understood in terms of catastrophe theory.

Physically, the validity of such an expansion is limited to the vicinity of the critical point. Landau's intuitive choice of the form of the thermodynamic potential functional waited for this mathematical justification for many years.

At this moment it is clear that this type of theory operates with averaged values which are the basic quantities of equilibrium thermodynamics. It is assumed that all the processes are taking place very slowly in such a way that the value of the thermodynamic variables is uniform in the whole sample.

There are some attempts to construct a catastrophe theory for systems containing many interacting subsystems but they are developed in the direction of the cellular automata theory [2].

To generalize the Landau theory one must include the energy of inhomogeneities. It is usually done by adding the gradient term (so called Ginzburg term) $(\nabla\eta)^2$. The appearance of this term is not justified mathematically, but seems to represent the desired phenomena very well and was successful in various models. As far as it was possible for me to trace it, for the first time such an extension of a thermodynamic order parameter was done by L.D. Landau and B.L. Ginzburg [39]. Despite the fact that the spatially dependent order parameter is used throughout this paper there is no discussion of the physical interpretation of this extension.

e) The coarse grained order parameter

Using partition function expressed in terms of the coarse-grained functional Eq. (1.4) the equilibrium order parameter $\bar{m}(\vec{r})$ in d-dimensional space can be calculated to be

$$\bar{m}(\vec{r}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{|\vec{k}|<k_c} dm_{\vec{k}} m(\vec{r}) \exp\left[\beta F - \int_{L^d} d^d\vec{r} F_L\right].$$

It is usually approximated by a function that minimizes the coarse grained functional F_L. Using the Euler equation one obtains

$$\frac{\delta F_L}{\delta m(\vec{r})} = 0. \tag{1.7}$$

A minimum of the thermodynamic potential which was obtained by calculation of the partial derivative, now is replaced by calculation of the functional derivative.

f) Renormalization group

On the other extreme of descriptions of large systems is the method of the renormalization group. It assumes that the correlation length is infinite at T_C and that the system looks the same on all length scales. Such conditions are usually satisfied at points of second order phase transitions. In the description of this method I follow L.E. Reichl [55].

The first step in the development of this theory was made by L.Kadanoff [27]. He proposed to divide a d-dimensional system of spins into subsystems containing L^d spins which is large compared to 1 but small compared to the number N of spins in the whole sample. Then he rewrote the original Hamiltonian

$$H(S) = -K \sum_{i,j}^{\gamma N/2} S_i S_j - B \sum_{i=1}^{N} S_i$$

(where S denotes a specific spin configuration, i and j numbers the lattice sites, K is an exchange constant, γ describes the number of nearest neighbours, and B corresponds to an external field), in terms of the new variables S_I. The new variables S_I can be defined as the average values of the old variables S_i over subsystems I. The new Hamiltonian takes the form

$$H(S_L) = -K_L \sum_{I,J}^{\gamma N L^{-d/2}} S_I S_J - B_L \sum_{I=1}^{N L^{-d}} S_I$$

This new block Hamiltonian looks exactly the same as the original one. Therefore one should expect to obtain the free energy per site $f(\epsilon, B)$ to be in the same functional form $f_L(\epsilon_L, B_L) = L^d f(\epsilon, B)$ where $\epsilon = (T - T_C)/T_C$ denotes deviation from critical temperature and B denotes the external field. It is clear that the new values of the free energy parameters are L dependent. L.Kadanoff assumed after B.Widom [64] that new values can be obtained by scaling $\epsilon_L = \epsilon L^x$, $B_L = BL^y$ where x, y are some constants. Knowing the behaviour of ϵ and B as functions of L one can calculate the power which dominates the divergence in the correlation functions. Calling $\lambda = L^d$ one gets

$$f(\epsilon, B) = \lambda^{-1} f(\lambda^p \epsilon, \lambda^q B) \tag{1.8}$$

where $p = -x/d$ and $q = -y/d$. Widom [64] expressed the critical exponents β and δ in terms of p and q

$$\beta = \frac{1-q}{p} \tag{1.9}$$

$$\delta = \frac{q}{1-q} \tag{1.10}$$

where β describes the divergence of magnetization at zero magnetic field and δ describes the divergence of magnetization along the critical isotherm. Using identities between critical exponents one can find most of the remaining critical exponents using the above two.

K.Wilson [65] extended Widom and Kadanoff's ideas. He wrote a model Hamiltonian in the form

$$H(\vec{K}, S, N) = K_0 + K_1 \sum_i S_i + K_2 \sum_{i,j}^{1} S_i S_j + K_3 \sum_{i,j}^{2} S_i S_j + \cdots$$

where $\sum^k$ means that only k^{th} neighbours are included. The coupling constants, K_i are temperature dependent. After summation over all blocks one obtains the equality of the two partition functions:

$$Z(\vec{K}, N) = Z(\vec{K}_L, NL^d)$$

Since the free energy per site has the same functional form as the old one it can be written:

$$f(\vec{K}) = \lim_{N\to\infty} \frac{1}{N} \ln Z(\vec{K}, N) = \lim_{N\to\infty} \frac{1}{N} \ln Z(\vec{K}_L, N) = L^{-d} f(\vec{K}_L)$$

The coupling constants are related by certain nonlinear transformation $\vec{K}_L = T(\vec{K})$. Repeating this operation many times for the critical point, at which correlations are infinite, one should reach a fixed point $\vec{K}^* = T(\vec{K}^*)$.

In the next step one investigates stability by calculating the eigenvalues and eigenvectors of the linearized map T around its fixed point $\vec{K}^*$. Rewriting the free energy in terms of eigenvectors $\delta\vec{u}$ one obtains

$$g(\delta u_1, \delta u_2, \cdots) = L^{-d} g(\lambda_1 \delta u_1, \lambda_2 \delta u_2, \cdots).$$

This expression has a similar form to Eq. (1.8) and it also provides a scaling relation for the free energy per site. Therefore, one can use the same ideas as before and express critical exponents in terms of the above scaling using Eqs. (1.9,1.10).

This theory has been very successful in finding critical exponents. However, to describe phenomena appearing on the mesoscopic scales one must go to the middle ground between thermodynamics and the renormalization group method.

g) Multicritical points

Multicritical points appear when lines of phase transitions of different order intersect. Landau's first paper [38] on phase transitions already contains a discussion of the critical point between the first and second order phase transitions.

To describe the behaviour of systems displaying multicritical points one often uses expansions containing higher order terms in the free energy expansion. For example, if one uses the expansion of the free energy:

$$\Phi(P, T, \eta) = \Phi_0 + \alpha\eta + A\eta^2 + C\eta^3 + B\eta^4 + D\eta^6$$

where $A(P,T) = 0$ but $B(P,T) > 0$, then the transition is of second order. However, if $B(P,T) < 0$ and $A(P,T) > 0$, then the transition is of first order. The point at which these two lines intersect is called *a multicritical point*. The graphical representation of Landau's discussion is presented in Fig. 1.2. In the space spanned by coefficients A and B one can see lines of first and second order transitions of the above free energy functional. The point at which the metastable states appear corresponds to $C = 0$ in Fig. 1.2. The point at which this metastable state becomes stable and the state $\eta = 0$ becomes unstable is denoted by C_c.

One may consider even more complicated geometries of the phase diagram as it was described by A.Aharony [3]. An example of a tricritical point taken from [3] is presented in Fig. 1.3. In the way characteristic to the Landau theory this diagram can be expanded introducing an additional variable H. The name of the tricritical point becomes clear in such extended space. Such expansion of a tricritical point is illustrated in Fig. 1.4. Also this figure is based on a sketch in [3]. High order expansions are often criticized, on the ground that terms corresponding to the powers higher than 4-th are irrelevant in the sense of the renormalization group theory. This argument does not take into account that one may be interested in the behaviour around phase transition points where the correlation length is still finite. Dropping the higher terms is correct in calculations of critical exponents but also limits the possible range of geometries of the mesoscopic structures. This thesis does not discuss the critical exponents but rather places importance on large scale structures and geometry of the phase diagram so the higher order terms are included in the following considerations.

h) Symmetry and Landau theory of phase transitions

In this section I will try to relate the above listed properties to the change of the symmetry group for a second-order phase transition. As stated in his first paper

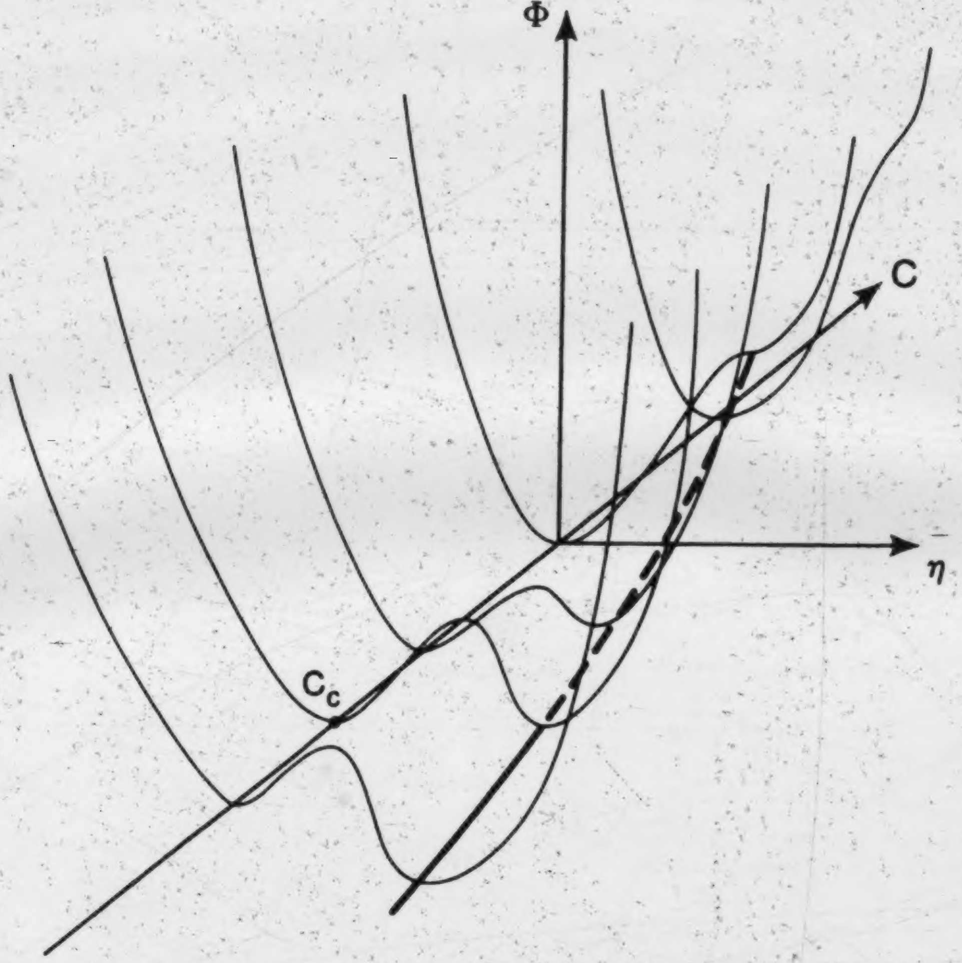

Figure 1.2: Schematic pictures of the first order phase transitions in Landau theory plotted in the C, η, Φ space. C_c denotes point at which the global minimum is different from zero

Figure 1.3 is removed due to copyrights restrictions.

Figure 1.3: The tricritical point O in a dilute magnet. The dashed curve represents the line of points of the second order phase transition, the continuous one the line of first order phase transitions.

Figure 1.4 is removed due to copyrights restrictions.

Figure 1.4: This diagram represents the tricritical point from Fig. 1.3 plotted in the larger space spanned by H, μ and T. The region limited by O, T_c, μ_c and lying in the $H = 0$ plane represents the first order transitions. Two "tricritical wings" represents the surfaces of the second order transitions.

[38] describing the idea of an order parameter L.D. Landau did not connect ideas of order parameter and symmetry. Two years later in 1937 he published another paper in two parts [36,37] in which he discussed phase transitions during which the symmetry group is changed.

If two phases have different symmetry properties any function, which is invariant under the action of an appropriate symmetry subgroup and is not invariant under the action of the whole group can be used as an order parameter. Landau has argued that it should be therefore possible to reconstruct his expansion starting from a discussion of the symmetry properties.

Reversing construction from previous sections one starts here from high- and low-symmetry states described by a density ρ and then writes the thermodynamic potential Φ in terms of these invariants. One assumes that the density ρ in the high-temperature phase and at the transition point itself has the symmetry group G_0 while in the low-temperature phase it has the symmetry group G. Here, G is a subgroup of G_0. The density function can be written as the sum $\rho = \sum_i \eta_i \phi_i$ where all the functions ϕ_i are transformed into combinations of one another by all transformations in the group G. We can decompose this sum into two parts the first of which is composed of the function which is invariant under all transformations of the group G_0 and the second part is not invariant under G_0.

$$\rho = \rho_0 + \delta\rho, \qquad \delta\rho = \sum_n {}' \sum_i \eta_i^{(n)} \phi_i(n)$$

where n numbers different representations and i numbers basis functions in each representation. In this formula the unit representation is excluded from the summation. The actual values of the $\eta_i^{(n)}$ as functions of P and T are determined thermodynamically from the conditions of equilibrium. It means that $\eta_i^{(n)}$ is found by minimization of the thermodynamic potential at given values of P and T. This determines the symmetry G of the crystal.

If the crystal is to have the symmetry G_0 at the transition point itself, it is necessary that all the $\eta_i^{(n)}$ should be zero there, i.e. $\delta\rho = 0, \rho = \rho_0$. Since the change in the state of the crystal in a phase transition of second order is continuous, $\delta\rho$ must tend continuously to 0, through arbitrarily small values near the transition point. Accordingly, one can expand the potential $\Phi(P, T, \eta_i^{(n)})$ in powers of $\eta_i^{(n)}$ near the transition point. Having fixed families of functions $\phi_i^{(n)}$ one can consider the representation of G acting on the space of the coefficients $\eta_i^{(n)}$ instead of the space of functions ϕ_i^n.

Next, since the thermodynamic potential is invariant under any of the transformations of the group G, the expansion can contain only invariant combinations of $\eta_i^{(n)}$ that are of the appropriate powers. No linear invariant can be formed from quantities which are transformed according to the non unit irreducible representation of a group, for otherwise that representation would contain the unit representation.

Thus, the leading terms in the expansion of Φ are of the form

$$\Phi = \Phi_0 + \sum_{(n)} {}' A^{(n)} \sum_i [\eta_i^{(n)}]^2 \tag{1.11}$$

where the $A^{(n)}$ are functions P and T.

At the transition point itself, the crystal must have the symmetry G, i.e. the equilibrium values of the $\eta_i^{(n)}$ must be zero. It is evident that Φ can have a minimum only when every $\eta_i^{(n)} = 0$ if all the $A^{(n)}$ are non-negative. If all the $A^{(n)}$ were positive at the transition point, they would also be positive near that point, so that the $\eta_i^{(n)}$ would remain zero and there would be no change of symmetry. For some $\eta_i^{(n)}$ to be non-zero, one of the coefficients $A^{(n)}$ must change sign, and this coefficient $A^{(n)}$ must therefore vanish at the transition point. (Two coefficients $A^{(n)}$ can vanish simultaneously at an isolated point in the PT-plane, which is the intersection of more than one line of transitions of second order).

The change in the sign of one of the $A^{(n)}$ causes the appearance of non-zero $\eta_i^{(n)}$ belonging to the n^{th} representation. Thus the crystal with symmetry G_0 becomes one with density $\rho = \rho_0 + \delta\rho$, where $\delta\rho = \sum_i \eta_i^{(n)} \phi_i^{(n)}$ is a linear combination of the base functions of any one of the irreducible representations of the group G_0 (other than the unit representation). Accordingly, the index n which gives the number of the representation, will be omitted meaning always the one which corresponds to the considered transition.

We shall use the notation: $\eta^2 = \sum_i \eta_i^2$, $\eta_i = \eta\gamma_i$ and write the expansion of Φ as

$$\begin{aligned}\Phi &= \Phi_0(P,T) + \eta^2 A(P,T) + \eta^3 \sum_\alpha C_\alpha(P,T) f_\alpha^{(3)}(\gamma_i) + \\ &\quad + \eta^4 \sum_\alpha B_\alpha(P,T) f_\alpha^{(4)}(\gamma_i) + \ldots,\end{aligned}$$

where $f_\alpha^{(3)}$, $f_\alpha^{(4)} \ldots$ are invariants of the third, fourth etc. orders formed from the quantities γ_i. In the sums over α there are as many terms as there are independent invariants of the appropriate order which can be formed from the γ_i. The previously listed conditions are imposed on the coefficients A, B, C and therefore the expansion up to the fourth order terms is in the form

$$\Phi = \Phi_0 + A(P,T)\eta^2 + \eta^4 \sum_\alpha B_\alpha(P,T) f_\alpha^{(4)}(\gamma_i)$$

In this way the basic results of Landau's first paper were recovered and a the connection between discrete symmetries and phase transitions were established. However, it was not until many years later that Landau included continuous symmetries in the framework of his theory [40]. He did it by adding the group of continuous translations to the already included discrete symmetries.

The representations of space groups can be indexed by the parameter $\vec{k}$ which takes a continuous series of values. The coefficients $A^{(n)}$ in Eq. (1.11) must therefore depend not only on the discrete number n but also on the continuous parameter $\vec{k}$.

Repeating the argument dealing with minimization of a thermodynamic potential that was used in the case of discrete symmetries one can argue that a phase transition should correspond to the vanishing (as a function of P and T) of the coefficient $A^{(n)}(\vec{k})$ with a given number n and a given $\vec{k} = \vec{k}_0$. In order that the transition should actually occur, it is necessary that $A^{(n)}$ as a function of $\vec{k}$ should have a minimum for $\vec{k} = \vec{k}_0$, i.e. the expansion of $A^{(n)}(\vec{k})$ in powers of $\vec{k} - \vec{k}_0$ should contain no linear terms.

The value of $\vec{k}_0$ determines the translational symmetry of the functions ϕ_i, and therefore that of the function $\delta\rho$, i.e. it determines the periodicity of the lattice of the new phase. The condition of stability at the transition point is translated into the condition that the structure at $\vec{k}_0$ must be stable in comparison with those which correspond to close values $\vec{k}_0$. But a structure with $\vec{k} = \vec{k}_0 + \vec{\kappa}$ where $\vec{\kappa}$ is small, differs from that with $\vec{k} = \vec{k}_0$ by a spatial modulation in the periodicity of the latter, that is, by the appearance of non-uniformity over distances $(1/\vec{\kappa})$ which are large compared with the periods of the lattice. At this point one can think about the thermodynamic potential as being dependent on the slowly varying coefficients η_i instead of being dependent on ϕ_i.

When η_i are space-dependent one has to include its derivatives in the expansion for the thermodynamic potential. One can neglect terms which are full derivatives. They disappear after the functional derivative is calculated, because using Stokes theorem they can be rewritten as integrals over the surface. Repeating the discussion on stability and on the invariance with respect to the action of discrete groups one can argue [40] that the derivatives should appear as combinations of squares of gradients of the order parameter.

i) Fluctuation of the order parameter

The main observation for this discussion is that when the system is in the equilibrium state η then the probability of the small deviation $\Delta\eta$ from the mean field solution η_0 is

$$w \propto \exp(-\Delta\Phi_t/kT) \tag{1.12}$$

where $\Delta\Phi_t$ denotes the change of the value of the thermodynamic potential Φ with respect to the equilibrium value, k is the Boltzmann constant and T the temperature.

For an inhomogeneous body it is better to use the thermodynamic potential $\Omega(\mu, T, \eta)$. To include the energy of inhomogeneities a quadratic form depending on derivatives of η

$$D_{ik}(\mu, T)\frac{\partial\eta}{\partial x_i}\frac{\partial\eta}{\partial x_k}$$

is added to the original functional. In the simplest case the matrix D may be taken as $D_{ik} = g\delta_{ik}$. For stability of a homogeneous body, D must be greater then zero and the density of the potential Ω can be written as

$$\Omega = \Omega_0 + \alpha\hat{t}\eta^2 + b\eta^4 + D(\vec{\nabla}\eta)^2 - \eta h \tag{1.13}$$

where α and b are taken per unit volume. The value of $\Delta\Omega$, in the linear approximation, is

$$\Delta\Omega_t = \int[\alpha\hat{t}(\delta\eta)^2 + D(\vec{\nabla}\delta\eta)^2]dV \tag{1.14}$$

The Fourier transform of the function $\delta\eta$ allows us to calculate the correlation functions for the reduced temperature $\hat{t} = T - T_C > 0$

$$<|\delta\eta_{\vec{k}}|^2> = \frac{T}{2V(Dk^2 + \alpha\hat{t})}$$

$$<\delta\eta_{\vec{r}_1}\delta\eta_{\vec{r}_2}> = \frac{T_c}{8\pi D r}\exp(-\frac{r}{r_c})$$

where $r_c = \sqrt{D/\alpha\hat{t}}$ is called the correlation length of the fluctuations.

The mean field solution for η is given by the minimum of the polynomial part of expression (1.13) which is $\eta^2 = 0$ for $\hat{t} > 0$ and $\eta^2 = -\alpha\hat{t}/b$ for $\hat{t} < 0$. We would like to determine when the mean square of the space average of the fluctuation is of the same order as the square of the phase over the volume corresponding to the correlation length. The space average of the fluctuation is given by the zero component of its Fourier transform, so its mean value was already calculated and it is $\mid \delta\eta \mid^2 = \frac{T}{2V\alpha t}$. Comparing these two quantities over the volume $V \propto r_c^3$ gives the Ginzburg criterion for the validity of Landau's expansion

$$\frac{T_c b^2}{\alpha D^3} \ll \mid \hat{t} \mid \tag{1.15}$$

j) *The continuous Ising model vs Landau expansion*

The following paragraphs describing the relation between the S^4 Ising model and the Landau theory are taken from the book by Patashinskii and Pokrovskii [48].

> A useful generalization of the Ising model is its continuum analogue. A varia[illegible] $\phi(\vec{x})$ ranging from $-\infty$ to ∞, is assigned to each site. The Hamiltonian of the system has the form
>
> $$H = \frac{I}{2}\sum_{\vec{x},\vec{b}}(\phi(\vec{x}) - \phi(\vec{x}+\vec{b}))^2 + \lambda\sum_{\vec{x}}(\phi(\vec{x})^2 - \phi_0^2)^2 \tag{1.16}$$
>
> As $\lambda \to \infty$ the system described by the hamiltonian H becomes isomorphic with the Ising model, i.e. in this limit $\phi(\vec{x})$ takes just two values $\pm\phi_0$. The model (1.16) is called the S^4 Ising model [55]. For any positive λ and I the ground state of the system is doubly degenerate $\phi(\vec{x}) = \pm\phi_0$, the sign being the same at all the lattice sites $\vec{x}$. In weakly excited states of the system $\phi(\vec{x})$ is slowly varying. In place of the lattice model (1.16) we introduce its continuum analogue:
>
> $$H' = \int \frac{c}{2}(\nabla\phi)^2 + \lambda'(\phi^2 - \phi_0^2)^2 d^d x$$

$$= \int \frac{c}{2}(\vec{\nabla}\phi)^2 + \frac{a}{2}\phi^2 + \lambda'\phi^4 d^d x + const \qquad (1.17)$$

where $c = Ib^{2-d}$, $a = -2\lambda\phi_0^2 b^{-d}$, $\lambda' = \lambda b^{-d}$, b is the lattice constant and d is the dimensionality of space.

By comparing Eq. (1.13) with Eq. (1.17) one can perceive the close analogy between the Landau theory and the continuous Ising model. The difference is that in the Ising and S^4 models the coefficients in the microscopic Hamiltonian are independent of the temperature. In Landau theory the coefficients are assumed to be functions of the temperature (one of the coefficients a changes sign as T passes through T_c).

1.2 Equation of motion

To include time dependent nonlinear relaxation processes M. Luban [43, page 71] proposed to add the time derivative to the Eq. (1.7)

$$\frac{\partial m}{\partial t} = -vR\frac{\delta F_L}{\delta m(\vec{r})} \qquad (1.18)$$

where R is a time-independent, positive, possibly temperature-dependent quantity with dimension of frequency and v is the volume per particle. Writing Eq. (1.18) one assumes that changes in m take place over times which are long compared to the relaxation times of the microscopic variables which are averaged over in order to define the order parameter. Otherwise, a hydrodynamic description of the system in terms of m would be invalid.

The rest of this section contains derivation of the equation of motion for a coarse grained macroscopic quantity. Derivation of this type was first made by J.S. Langer [41] in the case of the conserved order parameter. Extending Langer's ideas H.Metiu et al [44] derived an equation for the evolution of a nonconserved coarse grained macroscopic quantity. The starting point in this derivation is a Markovian master equation

$$\tau\frac{\partial P(x,t)}{\partial t} = -\sum_{\delta} W(x \rightarrow x+\delta)P(x,t) + \sum_{\delta} W(x-\delta \rightarrow x)P(x-\delta,t) \qquad (1.19)$$

where P(x,t) is the probability that the variable of interest (e.g. order parameter) has the value x at time t. The following assumptions are made about the system:

1. The system is characterized at all times by a free energy F.
2. Changes in the variables of interest (i.e. density distribution) which increase the free energy are less probable than those that decrease it.
3. On a given time scale τ, much shorter than the macroscopic time scale, large changes are very improbable.

Using these assumptions one can find the transition rate W [23]. If one considers the dynamics around the equilibrium position then the dynamics of a system should satisfy the detailed balance condition

$$W(x \rightarrow x+\delta)P_e(x) - W(x+\delta \rightarrow x)P_e(x+\delta) = 0.$$

Moreover, $P_e(x)$ should be proportional to the Boltzmann factor $\exp(-\beta F(x))$ [40]. Therefore,

$$\frac{W(x \rightarrow x+\delta)}{W(x+\delta \rightarrow x)} = \frac{P_e(x+\delta)}{P_e(x)} = \exp -\beta[F(x+\delta) - F(x)].$$

One can now write

$$W(x+\delta \rightarrow x) = \exp\{\frac{\beta}{2}[F(x+\delta) - F(x)]\}\Omega(x+\delta, x)$$

where Ω is a symmetric function of its two arguments. J.D. Gunton in his book [23] using the above assumption 3 argues that the function Ω in terms of the difference of its arguments should be sharply peaked around 0. H.Metiu et al [44] assume here a specific form of this function, namely $\Omega = \exp(-\frac{\delta^2}{\Delta})$ where Δ is a positive constant.

The above formalism can be extended to include interactions between many cells. Denoting by $n(x)$ the density of the variable at the point x one can rewrite the master equation (1.19) as

$$\begin{aligned}\frac{\partial P(n(x),t)}{\partial t} &= -\sum_{\delta(x)} W[n(x) \rightarrow n(x) + \delta(x)]P(n(x),t) + \\ &+ \sum_{\delta(x)} W[n(x) - \delta(x) \rightarrow n(x)]P[n(x) - \delta(x), t].\end{aligned} \tag{1.20}$$

Here, the intention is to go the infinite limit with division into small parts and changing summation into the functional integration. Using the Kramers-Moyal expansion [63, page 214] Eq. (1.21) can be rewritten as

$$\frac{\partial P(n,t)}{\partial t} = H(D,n)P(n,t) \tag{1.21}$$

where

$$H(D,n)P(n,t) = \sum_{\delta}\sum_{\alpha}\{\frac{(-1)^{|\alpha|}}{\alpha!}\}\delta^{\alpha}D^{\alpha}W(n \rightarrow n + \delta)P(n,t).$$

The formal solution of Eq. (1.21) is

$$P(n,t) = \exp[(t - t_0)H(D,n)]P(n,t_0).$$

Using the traditional construction of the Feynman path integral [20] this formula can be rewritten as

$$\begin{aligned}P(n,t) = \lim_{N\rightarrow\infty} &\exp[(t_N - t_{N-1})H(D,n_{N-1})]\delta(n_{N-1} - n_{N-2})\dots \\ &\delta(n_2 - n_1)\exp[(t_1 - t_0)H(D,n_0)]P(n,t_0)\end{aligned}$$

where $t_N = t$, $n_0 = n$. After Fourier transformation of δ functions and exponentials one can write the above operator as

$$\begin{aligned}&\int dn_{N-1}\cdots\int dn_1\int dk_{N-1}\cdots\int dk_0(2\pi)^{-mN} \\ &\exp[-\Delta t\sum_{j=0}^{N-1}\{< ik_j,(n_{j+1} - n_j)/\Delta t > -H(-ik_j,n_j\}].\end{aligned}$$

Evaluation of the above integral is simplified by limiting an infinite series in the Kramers-Moyal expansion to the first two nonzero terms. The result after performing integration over δ_j is

$$H(-ik,n) = -\frac{1}{2}\beta\Gamma\sum_{j=1}^{m} ik_j\frac{\partial F}{\partial n_j} - \frac{1}{2}\sum_{j=1}^{m} k_j^2, \tag{1.22}$$

with $\Gamma = (2\pi\Delta)^{m/2}\Delta$. The substitution into the evolution operator (1.22) and integration over k's gives

$$\exp[-\int_{t_0}^{t} dt' \frac{1}{2\Gamma}\sum_{j=1}^{m}(\frac{\partial n_j}{\partial t} + \frac{1}{2}\beta\Gamma\frac{\partial F}{\partial n_j})^2].$$

The integrand is positive so the only distributions that can yield a nonzero contribution are those for which the integrand is equal zero. Equating the integrand to zero one obtains equation of motion Eq. (1.18).

1.3 Experimental examples

In this section I would like to describe two experiments containing features that are interesting in the context of the symmetry reduction method. The first experiment shows a system which displays states with the continuous symmetry properties. The remarkable fact is that this system consists of crystalline thin film which does not have any continuous symmetries.

Quite recently in a series of papers, the Russian group [29] [30] reported very interesting behaviour of magnetic thin films under the influence of an external square-wave oscillating magnetic field. They reported the existence of large metastable structures appearing in the uniaxial iron-garnet $(YSm)_3(FeGa)_5O_{12}$ in the presence of a low-frequency $(10^2 - 10^4\ Hz)$ pumping. For very low frequencies and without an external field the usual labyrinthine domain structure was observed. Application of the static field leads to an increase in the size of domains orientated parallel to the field and later to saturation which takes place for fields in the range

$70 - 100\ Oe$. When the field oscillates with a frequency in the range $120 - 200\ Hz$ systems of concentric rings appear for a sample $5.5\mu m$ thick. The diameter of the internal ring is $200 - 400\ \mu m$ and that of the outer ring is $400 - 600\ \mu m$. The system of rings can move with a speed of approximately $10\mu m/sec$. The lifetime of such a system is of the order of $5 - 10\ sec$.

When the frequency is increased even more and takes values in the range $200 - 6000\ Hz$ one observes a system of spirals instead of rings. The core of the spiral is $20 - 30\ \mu m$ and the diameter of the whole spiral can reach up to $1\ mm$. The lifetime of the spiral depends on the number of turns it has; more turns implies a longer lifetime. It may reach $10\ sec$ for a frequency of $300\ Hz$. These spirals can move with a speed of $5 - 10\ Hz/sec$. Figure 1.5 represents the main features of the spiral pattern after A.S. Kandaurova et al [30].

Interestingly, for a $11.5\mu m$ thick sample rings patterns do not appear. Instead, spiral type domain boundaries appear starting at a frequency of $120\ Hz$. There are no visible patterns for frequencies above $6000\ Hz$ in both samples.

The most important features of this experiment, from our point of view, are

1. For frequencies in the range $120\ Hz < f < 200\ Hz$ either systems of rings (for the thin sample) or spiral domains (for the thick sample) appear
2. For frequencies $200\ Hz < f < 6000\ Hz$ spiral domains appear (for both samples)
3. There are no observed patterns for frequencies from $6\ kHz$ to $30\ kHz$
4. The above structures are 5 orders of magnitude larger compared to the lattice constant ($10\mathring{A}$)
5. The lifetime of the above structures is fairly substantial, i.e. up to the order of $10\ sec$

Figure 1.5 is removed due to copyrights restrictions.

Figure 1.5: A spiral pattern observed using the Faraday method in the magnetic thin film excited by an oscillating external electric field by A.S.Kandaurova et al, Soviet Physics JETP 70, (684), 1990

6. The above structures have ***continuous*** symmetry properties.

These experimental values illustrate how well the applicability conditions of the continuous approximation, as stated by Gunton et al [23], are satisfied in this case. These observations also indicate that the symmetry of the crystal lattice may not be preserved in the larger scale phenomena. It may be also possible that some role in symmetry considerations is played by the fact that one deals here with a very thin, almost two dimensional structure. It is not quite clear what would happen in three-dimensions. This aspect is manifested through the lack of ring domain boundaries for the thick sample.

The second system consists of a molecular fluid. Therefore, one can expect both continuous and discrete symmetries to appear. The interesting feature of this system is the system of patterns appearing when the value of the control parameter is changed.

Another example of self organization in systems driven by an external field is investigated by the French group of R. Ribotta and A. Joets [57][26]. In this experiment a thin layer (approximately $50\mu m$) of a liquid crystal is excited by an external oscillating electric field with frequency around 150 Hz. When the amplitude of the electric field applied perpendicularly to the plane of the liquid crystal is less then 7 V the observed pattern is called ***Normal Rolls*** and it consists of parallel rolls of the same thickness as the whole layer. A schematic diagram of the experimental setup in the case of the Normal Rolls is presented in Fig. 1.6 An increase of the amplitude above the critical value of 9 V leads to a change of the pattern to so-called ***Oblique Rolls***. The next threshold is approached at the amplitude value of 11 V when so called ***Skewed-Varicose Rolls*** appear. Then at 15 V the more complicated ***Bimodal*** structures appear. A further increase of the applied field leads to an apparently chaotic behaviour of the system. This sequence is illustrated in Fig. 1.7.

Figure 1.6 is removed due to copyrights restrictions.

Figure 1.6: The experimental setup of the experiment by A. Joets et al Physica 23D (1986); AC electric field is applied in the direction **z**

Figure 1.7 is removed due to copyrights restrictions.

Figure 1.7: Examples of patterns obtained by A. Joets et al in liquid crystals; taken from A. Joets et al Physica 23D (1986) 235-239

The above described series of transitions is characteristic of the slow changes in amplitude (slower than 20 mV/min) but it may be significantly different for rapid changes of amplitude. In such a case the obtained patterns contain some defects. These defects have the symmetry properties related to those of the phase corresponding to the higher value of control parameter, i.e. the amplitude of the driving field. Therefore, they seem to mediate the transition to the higher phase.

The most interesting properties of this system are

1. Patterns are extended in the whole sample that is over distances large compared with the distances between nematic molecules
2. The symmetry of patterns is decreased as the value of the control parameter is increased.
3. The path of phase transformations depends on the rate of change of the amplitude of the driving field.
4. There is a well defined sequence of symmetries appearing in the system

It is important to realize that both the above systems are studied under nonequilibrium conditions. It seems that different patterns are sustained by the energy supplied by the driving field.

1.4 The time- and space-dependent Landau-Ginzburg equation and its modification with mixed signature

As was explained above the actual form of the free energy functional usually represents an educated guess because it is very difficult to derive it by rigorous statistical means starting from microscopic variables. It is usually quite plausible to accept the original form of the Landau-Ginzburg equation. This equation has the internal structure which is rich enough to include most of the interesting phenomena, and at the same time is reasonably simple.

$$F[M] = \int [A_0 + A_1 M + \frac{1}{2} A_2 M^2 + \frac{1}{4} A_4 M^4 + \frac{1}{6} A_6 M^6 + \frac{1}{2} \sum_{\mu=x,y,z} D_\mu (\frac{\partial M}{\partial \mu})^2]\, d^3 r. \quad (1.23)$$

The standard physical interpretation of this functional is as follows:

A_0 : This term corresponds to the part of the free energy density which is independent of the order parameter M.

$A_1 M$: This term usually describes the interaction of the order parameter with its conjugate force A_1. For a magnetic system A_1 would be equal to the negative external magnetic field.

the quadratic, quartic and sextic terms : These terms usually describe the average local interaction with the local field which is proportional to the polynomial in the odd powers of M.

the gradient term : This term is responsible for the energy of inhomogeneities.

The coefficients D_μ appearing in the functional (1.23) describe the strength with which the order parameter interacts with itself in different directions. Usually all the coefficients D_μ are positive because this is necessary to guarantee the stability of the minimum with respect to small perturbations.

However, we would like to explore a slightly more general form and allow some of the coefficients D_μ to be negative. In order to satisfy physical requirements we are forced to consider some cutoff conditions necessary to limit the value of the free energy functional from below. However, one has to remember that the cutoff condition related to the coarse graining was already introduced into our considerations; oscillations with the wavevector larger then k_c were already eliminated. It does not, therefore, appear to be a serious limitation.

The usual cutoff procedure is related to the fact that the smallest length scale appearing in the problem is the size of the unit cell of the crystal. There should not be any variables which vary over smaller distances. Another condition limiting the class of solutions is that the amplitude should be limited by some physically chosen maximum of the respective quantity. For example when one deals with magnetization there is a fairly obvious limit to the magnitude of the magnetic moment which is realized when all possible spins are parallel.

Additionally, we are interested in the possibility of reentrant phase transitions when the coefficients D_μ are temperature dependent. This problem will be discussed more extensively in section (3.3). The physical interpretation of the negative coefficients D_μ is that it corresponds to the antiparallel orientation of the order parameter in planes perpendicular to the direction μ.

Applying Eq. (1.18) to Eq. (1.23) one can obtain an equation of motion for the Landau-Ginzburg type of the free energy functional. First, however, one has to evaluate its functional derivative through

$$\begin{aligned} F[M &+ \delta M] = F[M] + \\ &+ \int_V (A_1 + A_2 M + A_4 M^3 + A_6 M^5 - \sum_{\mu=x,y,z} D_\mu \frac{\partial^2 M}{\partial \mu^2}) \delta M d^3 r \quad (1.24) \\ &+ \int_{\partial V} \sum_{\mu=x,y,z} (\frac{\partial M}{\partial \mu})(\frac{\partial \delta M}{\partial \mu}) + O(\delta M^2). \quad (1.25) \end{aligned}$$

At this point one usually assumes that the surface term described in Eq. (1.25) vanishes. It should be pointed out, however, that such an assumption has to be deduced from the physics of the model and is by no means trivial. Hereafter, it is assumed that these terms are equal to zero. It is equivalent to assuming that the variational principle is applied to the problem with fixed values at the boundary. It can be accomplished by two different assumptions. The first one is that there is an open neighbourhood of the boundary of the domain V in which every solution is constant. Alternatively, one would assume that this condition is forced on variations of the functional F. Physically, it means that the value of the variation δM at the boundary is fixed.

Now one can use the above value of the functional derivative and obtain the basic equation of this thesis:

$$\frac{\partial M}{\partial t} = \sum_{\mu=x,y,z} D_\mu \frac{\partial^2 M}{\partial \mu^2} - (A_1 + A_2 M + A_4 M^3 + A_6 M^5). \qquad (1.26)$$

where the t variable was rescaled to include coefficients appearing in Eq. (1.18).

Using the substitution $x = \sqrt{|D_x|}x'$, $y = \sqrt{|D_y|}y'$, $z = \sqrt{|D_z|}z'$, $t = \epsilon_t t'$, $a = \epsilon_1 A_1$, $b = \epsilon_2 A_2$, $c = \epsilon_4 A_4$, $d = \epsilon_6 A_6$ where $k = 1,2,4,6$ and all $\epsilon = \pm 1$ and dropping all primes one can transform equation (1.26) to its canonical form:

$$\frac{\partial M}{\partial t} + \frac{\partial^2 M}{\partial x^2} + \epsilon(\frac{\partial^2 M}{\partial y^2} + \frac{\partial^2 M}{\partial z^2}) = a + bM + cM^3 + dM^5 \qquad (1.27)$$

The above equation includes two possible cases corresponding to different signs of the $\epsilon = \pm 1$ constant. The symmetry reduction of the $\epsilon = 1$ case is discussed in section 3.1 while the case corresponding to $\epsilon = -1$ is discussed in section 3.2.

After this rather lengthy discussion of the coarse-grained free energy one may be led to thinking that there is a unique way of interpretation of the free energy functional. The success of the application of the Landau-Ginzburg free-energy functional makes many researchers believe that its interpretation can be extended and that it can be used to treat variations of the order parameter which take place on the atomic length scale. It is not quite clear how this can be justified in more fundamental terms. One of the examples of such a procedure is given in the paper by A. Michelson [45].

Another example was given by W. Khan [31]. In this paper he attempts to explain the periodic noncommensurate magnetic ordering in the RM_2Si_2 alloy in a certain range of temperatures where R is the rare earth element and M is a transition element. He suggested that for the sinusoidal modulation of the magnetic moment in $PrNi_2Si_2$ whose period is slightly above 2 lattice constants and which takes place below $T_N = 18\ K$, the problem can be adequately treated by solving the variational equation of the free energy functional.

Many results presented here can be applied on the very small length scales if this approach is correct. The more fundamental arguments lead us, however, to believe that results presented in chapter 3 are justified for slowly varying solutions.

CHAPTER TWO

THE METHOD

In this chapter I will give an outline of the mathematical method which is used throughout this thesis. It starts from the symmetry reduction method in general. An example of the application of this method to the 1+1-dimensional heat equation follows. The last section of this chapter contains a discussion of the Painlevé test and property. The main definitions and theorems are given in the appendix to make this thesis self-contained and easier to read at the same time.

2.1 The symmetry reduction method

This section is devoted to a brief introduction of the method of symmetry reduction. I shall assume a basic knowledge of differential geometry and Lie group theory.

The following introduction to the method of the symmetry reduction was proposed by G.W. Bluman and J.D. Cole [8].

Consider a partial differential equation in independent variables x, t and dependent variables u

$$\Delta(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}) = 0. \tag{2.1}$$

Together with m associated boundary conditions it will be called S. Consider a one-parameter Lie group of transformation (parametrized by ϵ):

$$x^* = X^*(x, t, u; \epsilon) \tag{2.2}$$

$$t^* = T^*(x, t, u; \epsilon) \tag{2.3}$$

$$u^* = U^*(x, t, u; \epsilon) \tag{2.4}$$

Say $u = \Theta(x, t)$ is a solution to the system S (2.1). Consider the new system S^*

obtained from S by substitution

$$x \rightarrow x^*$$
$$t \rightarrow t^*$$
$$u \rightarrow v$$

One would say that the system S is invariant with respect to the group action if and only if $v = U^*(x,t,\Theta(x,t))$ satisfies S^* whenever $u = \Theta(x,t)$ satisfies S.

If one assumes that solutions $v = U^*(x,t,u;\epsilon)$ and

$$u = \Theta(X^*(x,t,u;\epsilon), T^*(x,t,u;\epsilon))$$

are equal then equating terms in different orders of ϵ one may find the invariant solution Θ. Expansion to the first order in ϵ of the formulae (2.3 - 2.4) gives

$$\begin{aligned} x^* &= x + \epsilon\zeta(x,t,u) + O(\epsilon^2) \\ t^* &= t + \epsilon\tau(x,t,u) + O(\epsilon^2) \\ u^* &= u + \epsilon\eta(x,t,u) + O(\epsilon^2). \end{aligned}$$

Which in turn leads to

$$\begin{aligned} \Theta(x + \epsilon\zeta(x,t,u) + O(\epsilon^2), t + \epsilon\tau(x,t,u) + O(\epsilon^2)) = \\ = \Theta(x,t) + \epsilon\eta(x,t,u) + O(\epsilon^2). \end{aligned}$$

In the limit $\epsilon \rightarrow 0$ one obtains the first order linear partial differential equation for Θ

$$\zeta(x,t,\theta)\Theta_x + \tau(x,t,\theta)\Theta_t = \eta(t,x)$$

which usually can be solved yielding $\Theta = F(x,t,f(\xi))$.

The above procedure assumes the knowledge of the action of the one-parameter symmetry group. Therefore one needs a method which would allow determination of the appropriate group. In general it is easier to look for the associated Lie algebra than for the Lie symmetry group. In the process of determining this symmetry it is also more convenient to consider a differential equation as an algebraic equation defined on the space of independent and dependent variables as well as all its derivatives.

The main idea of the symmetry reduction method can be explained using a simpler analogous procedure applied to a single algebraic equation defined by a function Δ

$$\Delta(x, u^{(n)}) = 0 \tag{2.5}$$

where $u^{(n)}$ denotes all derivatives of u with respect to x up to the order n. In the case of algebraic equations all derivatives of u are treated as independent variables. Let Δ be a function acting from a subset $\Omega \subset R^p$ into R^q. The interesting objects are *level sets* of the function Δ $\{x \in R^p | \Delta(x) = c\}$ for any fixed $c \in R^q$ of the function Δ. Under some assumptions on the continuity class of the function Δ and on the behaviour of its derivative (the implicit function theorem) each level set is a submanifold of Ω.

Using symmetry transformations one can start from any point of any level set and remain in the same level set for some values of the parameter ϵ which are close enough to zero (see Definitions A.1 and A.2). It has to be pointed out that one can find a much larger symmetry group for a particular level set than for all of them combined.

The condition on the transformation group which assumes equality of the transformed solution and of the solution at the transformed point described by G. Bluman [8] is called the group invariance. A few generalizations of this concept are given in Definitions A.3,A.4 and A.5.

The above ideas can be generalized to a case where the function f is a differential equation. Points which belong to the zero level set of the function Δ are solutions of algebraic equation (2.5) and therefore analysis of the symmetry group for solutions is equivalent to finding a local group of transformations of the zero level set. The corresponding definition describing a solution is A.6.

The main difference between a differential equation and a simple function is that the variables appearing as the arguments of the differential equation are not independent. The action of the group may depend only on x, u but it may also depend on some derivatives of u. In this thesis only the first case is discussed. The action of the group on the derivatives can be obtained assuming a covariant change of derivatives (point symmetries). The appropriate formula is called a prolongation of the action of the group (see Definition A.7). It is easier to calculate the prolongation formula for the associated Lie algebra then for a Lie group.

To actually use this method one has to know how to calculate the $n-th$ prolongation of a vector field. The general formulae for the prolongation of the vector field which are presented below are taken from [47].

Let the vector field v be given by

$$v = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x,u)\frac{\partial}{\partial u^\alpha}.$$

Note that since the coordinates $(x, u^{(n)})$ on $M^{(n)}$ consist of the independent variables $(x^1, \dots, x^p)$ and all derivatives u_J^α of the dependent variables up to the order n, a vector field on $M^{(n)}$ will in general take the form

$$\hat{v} = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_J \phi_\alpha^J \frac{\partial}{\partial u_J^\alpha}$$

the latter sum ranging over all multi-indices J of order $0 \leq \mid J \mid \leq n$; the coefficient functions ξ^i, ϕ_α^J could depend on all the variables $(x, u^{(n)})$. In terms of coefficients of the given vector field v, $pr^n(v)$ is written as

$$pr^n(v) = v + \sum_{\alpha=1}^{q} \sum_J \phi_\alpha^J(x, u^{(n)}) \frac{\partial}{\partial u_J^\alpha}$$

where $1 \leq | J | \leq n$ and the coefficient functions ϕ_α^J of $pr^{(n)}v$ are given by the following formula:

$$\phi_\alpha^J(x, u^{(n)}) = D_J(\phi_\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha) + \sum_{i=1}^{p} \xi^i u_{J,i}^\alpha,$$

where $u_i^\alpha = \partial u^\alpha / \partial x^i$, $u_{J,i}^\alpha = \partial u_J^\alpha / \partial x^i$, and the action of the total derivative D_i on a function $P(x, u^{(n)})$ is defined as

$$D_i P = \frac{\partial P}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{J} u_{J,i}^\alpha \frac{\partial P}{\partial u_J^\alpha}$$

After introducing the basic concepts of the symmetry reduction method the outline of this method can be written as follows. A system S of n-th order differential equations in p-independent and q-dependent variables is given as a system of equations $\Delta_\nu(x, u^{(n)}) = 0, \nu = 1, \ldots, l$ involving $x = (x^1, \ldots, x^p)$, $u = (u^1, \ldots, u^q)$ and the derivatives of u with respect to x up to order n. The function Δ can be viewed as a smooth map from the jet space $X \times U^{(n)}$ to some l-dimensional Euclidean space, $\Delta : X \times U^{(n)} \rightarrow R^l$. From this point of view, a smooth solution of the given system of differential equations is a smooth function $u = f(x)$ such that $\Delta_\nu(x, u^{(n)}) = 0, \nu = 1, \ldots, l$, whenever x lies in the domain of f.

Now suppose G is a local group of transformations acting on an open subset $M \subset X \times U$ of the space of independent and dependent variables. There is an induced local action of G on the n-jet space $M^{(n)}$, called the **n-th prolongation of G** and denoted $pr^{(n)}G$. This prolongation is defined so that it transforms the derivatives of function $u = f(x)$ into the corresponding derivatives of the transformed function $\tilde{u} = \tilde{f}(\tilde{x})$.

Combining Theorems A.10 and A.11 we can transform the condition involving invariance of S_Δ under the action of the prolongation of elements of the group G into a condition involving prolongation of infinitesimal generators v of G (see Theorem A.12). Theorem A.13 shows that under local solvability condition also the reverse implication holds. This extra condition has to be introduced because there is no exact equivalence between the set of solutions of the system S understood as an algebraic system and the set of solutions of the same system understood as a system of differential equations S. For a given point $(x, u^{(n)})$ which belongs to S_Δ a corresponding function does not have to exist. To deal with such a pathology we introduce the local solvability condition which is intimately connected to the Cauchy-Kovalevskaya Theorem and is included in the nondegeneracy condition.

Theorem A.13 gives us a practical way to look for transformation groups of a system of nonlinear differential equations. Let us assume that the vector field v is an infinitesimal generator of the symmetry group of the equation Δ. We can then calculate the $n-th$ prolongation of v, in terms of the yet unknown coefficients of v, and apply it to Δ. The resulting equation can be simplified using the condition $\Delta(x, u^{(n)}) = 0$. The system of first-order partial differential equations for the coefficients of v is obtained by inspecting coefficients of all independent and dependent variables together with derivatives of the latter ones. We were able to solve this system of of equations in our cases and in this way we are able to find the symmetry group associated with the equation Δ.

The first part of this problem, namely the construction of prolongation and obtaining the determining system of equations is a completely algorithmic procedure. It can be performed by a computer using a symbolic calculation language. Our calculations were done at the Centre des Recherches Mathematiques, Universite de Montreal. A program written in the symbolic language MACSYMA by Champagne and Winternitz [11] was used.

To proceed further, we have to discuss the existence of invariants related to the previously constructed algebra g.

In practice, the s-dimensional orbits are defined by the appropriate subalgebra of the associated algebra g. Let us take a subalgebra $g_0 \subset g$ where g is the associated algebra of the symmetry group G. The vector fields generating the subalgebra g_0 are in involution and they satisfy the assumptions of the Frobenius theorem. Therefore, they define a family of completely integrable submanifolds in the space of independent and dependent variables.

a) Reductions of the system of partial differential equations Δ

The next step in the analysis is finding solutions of the system of PDE's Δ knowing its symmetry group. One method which works in the case of a 1+1-dimensional second order equation was already presented above after [8]. A description of a more general approach is given below.

Consider a system of partial differential equations Δ defined over an open subset $M \subset X \times U \simeq R^p \times R^q$ of the space independent and dependent variables. Let G be a local group of transformations acting on M. The solution $u = f(x)$ is said to be G-invariant if it is left unchanged on the common domain by all the group transformations in $g \in G$ such that $g \cdot f$ is defined.

If G is a symmetry group of a system of partial differential equations Δ, then, under some additional regularity assumptions on the action of G, we can find all the G-invariant solutions to Δ by solving a reduced system of differential equations, denoted by Δ/G, and then writing these solutions in terms of independent variables of the original system Δ.

We make the regularity assumption that both the action of G on M and the projected action of G on Δ/G are regular, and that the orbits of the latter one have dimension s, where s is strictly less than p, the number of independent variables in the system. The case $s = p$ is fairly trivial, while for $s > p$ there are no G-invariant solutions.

Under those assumptions there are $p + q - s$ invariants η, ζ. The system Δ

can be now expressed in terms of those invariants and some parametric variables. The next regularity assumption is that it is possible to recover the old variable u out of variables η, ζ. To assure this it is assumed that the last q columns of the Jacobian matrix J have rank q everywhere

$$rank(\partial\eta^i/\partial u^\beta, \partial\zeta^\alpha/\partial u^\beta)^T = q$$

It should be mentioned that when the generators of the action of the group G are independent of the old dependent variables u then the new dependent variables ζ can be chosen as $\zeta = u$.

The last remaining problem with the quotient structure is that some submanifold given by the Frobenius theorem may not represent any function because the derivative of the dependent variables with respect to independent variables may go to infinity. Such submanifolds would correspond to solutions on the smaller domain.

In practice, the s-dimensional orbits are defined by the appropriate subalgebra of the associated algebra g. Let us take a subalgebra $g_0 \subset g$ where g is the associated algebra of the symmetry group G. The vector fields generating the subalgebra g_0 are in involution and they satisfy the assumptions of the Frobenius theorem. Therefore, they define a family of completely integrable submanifolds in the space of independent and dependent variables.

b) Analysis of the algebraic structure of Lie algebras

In the second part I will describe the basic principles of classification of subalgebras and finding the representatives of conjugacy classes. There are many subalgebras that generate reductions. The next problem is to classify them in a convenient way. The appropriate idea comes from the Lie Group theory (Theorem A.15) and it can be translated into corresponding subalgebras using the following theorem (Theorem A.16).

There is no unique description of the structures of subalgebras as it can be easily seen by applying the adjoint transformation to the entire system at once. The result of such an operation transforms each representative into the corresponding conjugated representative. However, the physically important characteristics remain invariant under such operations. For example, the dimensions of subalgebras, the dimensions of orbits, and the inclusion structure are preserved under the action of an adjoint transformation.

Methods of analyzing the structure of systems of subalgebras were developed in a series of papers by J. Patera, R.T. Sharp, P. Winternitz and H. Zassenhaus [53,52,50,51,49]. In these papers they indicated the method of a systematic determination of all possible subclasses of subalgebras. Their method is based on the decomposition of an algebra into the semisimple product of its two subalgebras. Sometimes one can use the Levi-Malcev decomposition which represents an algebra as the semisimple product of a radical and semisimple algebra.

It is natural to divide subalgebras into two types: splitting and nonsplitting. The splitting algebras are generated by generators that are elements of the radical or the semisimple part of the decomposition. They do not contain generators that are sums of elements from two components of the decomposition. The nonsplitting algebras contain at least one generator that mixes two elements from different parts of the semidirect decomposition. For this rather technical reason these two types are distinguished in Tables 2 and 7 by writing a tilde above the subalgebra notation for that of the nonsplitting type.

c) *Example. The 1+1-dimensional heat equation*

In the following I would like to illustrate the main steps of determining the Lie algebra and then discuss the results of symmetry reduction applied to the 1+1 dimensional linear heat equation. The details of this calculations are given in P.Olver's book [47, page 120]

The 1+1-dimensional heat equation can be written in the form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{2.6}$$

is a linear second order PDE containing a first derivative term with respect to time t and a second order derivative term with respect to position x.

The generator of the Lie algebra associated with Eq. (2.6) can be represented as a vector field and as such can be written in the form

$$v = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u}.$$

Then its second prolongation pr^2 is

$$pr^2 v = v + \phi^x \frac{\partial}{\partial u_x} + \phi^t \frac{\partial}{\partial u_t} + \phi^{xx} \frac{\partial}{\partial u_{xx}} + \phi^{xt} \frac{\partial}{\partial u_{xt}} + \phi^{tt} \frac{\partial}{\partial u_{tt}}$$

where functions required in the following calculations are [47, page 117]

$$\phi^t = \phi_t - \xi_t u_x + (\phi_u - \tau_t)u_t - \xi_u u_x u_t - \tau_u u_t^2$$

$$\begin{aligned} \phi^{xx} \;=\; & \phi_{xx} + (2\phi_{xu} - \xi_{xx})u_x - \tau_{xx}u_t + (\phi_{uu} - 2\xi_{xu})u_x^2 \\ & -2\tau_{xu}u_x u_t - \xi_{uu}u_x^3 - \tau_{uu}u_x^2 u_t + (\phi_u - 2\xi_x)u_{xx} \\ & -2\tau_x u_{xt} - 3\xi_u u_x u_{xx} - \tau_u u_t u_{xx} - 2\tau_u u_x u_{xt}. \end{aligned}$$

Applying the second prolongation $pr^2 v$ to Eq. (2.6) one finds that the infinitesimal criterion of invariance is $\phi^t = \phi^{xx}$ which must be satisfied whenever $u_t = u_{xx}$. Using the explicit form of the functions ϕ^t and ϕ^{xx} one finds an equation containing a variety of combinations of derivatives of the dependent variable u. The system of determining equations are obtained by equating the coefficients of all derivatives of u to zero. The solutions of the determining equations are the following

$$\begin{aligned} \xi &= c_1 + c_4 x + 2c_5 t + 4c_6 xt \\ \tau &= c_2 + 2c_4 t + 4c_6 t^2 \\ \phi &= (c_3 - c_5 x - 2c_6 t - c_6 x^2)u + \alpha(x,t) \end{aligned}$$

where $c_1, \ldots, c_6$ are arbitrary constants and $\alpha(x,t)$ is an arbitrary solution of the heat equation. The Lie algebra is spanned by the six vector fields

$$\begin{aligned} v_1 &= \partial_x \\ v_2 &= \partial_t \\ v_3 &= u\partial_u \\ v_4 &= x\partial_x + 2t\partial_t \\ v_5 &= 2t\partial_x - xu\partial_u \\ v_6 &= 4tx\partial_x + 4t^2\partial_t - (x^2 + 2t)u\partial_u \end{aligned}$$

and the infinite-dimensional subalgebra generated by $v_\alpha = \alpha(x,t)\partial_u$ where α is an arbitrary solution of the heat equation.

The first two generators reflect the fact that there is no distinguishable frame of reference for Eq. (2.6). The generator v_3 reflects the fact that one can multiply a solution by an arbitrary number. The generator v_α reflects additivity of solutions which is the second part of the linear property of Eq. (2.6). The remaining generators describe the scaling property (generator v_4) and a kind of Galilean transformation (generator v_5).

Calculating the action of the one-parameter group generated by v_6 one finds

$$(x,t,u) \xrightarrow{\exp(\epsilon v_6)} (\bar{x}, \bar{t}, \bar{u})$$

$$\tilde{x} = \frac{x}{1-4\epsilon t},$$
$$\tilde{t} = \frac{t}{1-4\epsilon t},$$
$$\tilde{u} = u\sqrt{1-4\epsilon t}\exp(\frac{-\epsilon x^2}{1-4\epsilon t})$$

which for a constant solution $u(x_0, t_0) = c$ takes the form

$$u(\tilde{x}, \tilde{t}) = \frac{c}{\sqrt{1+4\epsilon\tilde{t}}}\exp(\frac{-\epsilon\tilde{x}^2}{1+4\epsilon\tilde{t}})$$

The last formula can be interpreted in the following way: if one knows the initial condition for a point that belongs to a generic orbit, then one can calculate its influence on the value of solutions at other points. The final result is obtained through integration over the domain of influence due to linearity of Eq. (2.6).

2.2 The Painlevé property

This part is written on the basis of the book by Drazin [15] and the paper by Ablowitz et al [1]. Solutions of an ordinary differential equation may have singularities of different types (poles, branch points, essential singularities). Those of them which depend on integration constants are called movable singularities. However, there are equations for which the position of critical points (singularities other than poles) do not change when initial conditions are changed. It is said that in such a case the equation possess the Painlevé property.

For the ordinary differential equations of the first and second order there is a complete list of such equations under some limitations on the form of the equation. Namely, the right hand side of the equation under consideration must be a rational function of a dependent variable and an analytic function of an independent variable.

There are 50 types of second order differential equations having the Painlevé property. Solutions of these equations are expressed in terms of elementary functions, elliptic functions and the six Painlevé transcendents. The list of these equations with their solutions can be found in the book by E.Ince [25].

Therefore, it is convenient to know if a given ordinary differential equation has the Painlevé property because when it does then it is possible to look for an appropriate transformation to one of the listed equations. It is a way to look for analytic solutions.

There is a computationally feasible algorithmic test to check the necessary condition for an equation to have the Painlevé property. It was constructed by Ablowitz et al [1]. The Painlevé test was constructed with the intention to determine whether an equation can be solved by the inverse scattering method. This motivation is irrelevant to the way it was used in this work.

In the first step of the Painlevé test it is determined what is the lowest power in the Laurent series expansion of a solution around an assumed singularity. It is done by assuming that a solution $w(z)$ is proportional to $\alpha(z-z_0)^p$ and calculating the lowest power p for which this term is eliminated from the ODE. This term in the expansion is called ***the leading term***. The power p which appears in the leading term should be a negative integer because only then the singularity can be a pole. If p is a negative rational number then through an appropriate substitution the equation can be transformed to one with an integer p.

The next step is to investigate the relation of the leading term to next terms in the expansion. It is done by adding the next term to the leading term in the expansion $w(z) = \alpha(z-z_0)^p + \beta(z-z_0)^{p+r}$. Here r should a positive integer. These positive integers r are called ***resonances***. These relations must be satisfied for any integration constants obtained from an inductive integration of the equation using only the first few terms in the expansion.

In the last step one checks if coefficients in the expansion can be expressed in terms of the constants of integration by substituting $w(z) = \alpha(z - z_0)^p + \sum_{j=1}^{r} a_j(z - z_0)^{p+j}$. It should be stressed at this point that this method allows one to check only the necessary condition for the equation to be of Painlevé type. There are examples of equations satisfying the Painlevé test and having movable essential singularities.

The above test can be written as an algorithmic procedure and is in practice executed by the computer. We used programs written in the symbolic language MACSYMA by Rand and Winternitz [54] and by W.Hereman [24].

CHAPTER THREE

RESULTS

3.1 The Landau-Ginzburg equation with positive signature of derivatives

This chapter contains the results of symmetry reduction analysis of the partial differential equation (1.27) where $\epsilon = 1$.

$$\frac{\partial M}{\partial t} + \frac{\partial^2 M}{\partial x^2} + \frac{\partial^2 M}{\partial y^2} + \frac{\partial^2 M}{\partial z^2} = a + bM + cM^3 + dM^5 \tag{3.1}$$

Different aspects of the symmetry reduction are discussed in the following order

1. Algebras
2. Systems of subalgebras
3. The geometry of ordinary differential equations (ODE's)
4. Algebraic and first order equations
5. Results of the Painlevé test and solutions of some ODE's

These results are original and were published in [60] and [59]. It should be mentioned, however, that some earlier results obtained by the means of symmetry reduction and describing time-independent solutions were given by J.Rendell [56] and in the time-independent case where the right hand side polynomial was limited to a 3^{rd} order expansion by P.Winternitz et al [66].

a) The Lie algebras of the Landau-Ginzburg equation

The determining equations for Eq. (3.1) were obtained using the program written by B.Champagne et al [11] in the symbolic language $MACSYMA^{TM}$. The second independent check was done using a program written by A.K.Head in the symbolic language $MUMATH^{TM}$. Generators of the algebras were calculated by hand using the determining equations obtained in the previous step. These results were checked using the above mentioned program written by A.K. Head.

There are three possible symmetry groups for equation (3.1) when [illegible] least one of the nonlinear terms is different form zero.

1. $d \neq 0$ and $a = b = c = 0$,
2. $c \neq 0$ and $a = b = d = 0$,
3. all other possibilities (where it is implicitly assumed that the equation has to remain nonlinear; i.e. c and/or d are not equal 0)

Using the following generators

Translations $P_x = \frac{\partial}{\partial x}$, $P_y = \frac{\partial}{\partial y}$, $P_z = \frac{\partial}{\partial z}$, $P_t = \frac{\partial}{\partial t}$

Rotations $L_x = zP_y - yP_z$, $L_y = xP_z - zP_x$, $L_z = yP_x - xP_y$

Dilatations $D_1 = xP_x + yP_y + zP_z + 2tP_t - M\frac{\partial}{\partial M}$,

$D_2 = xP_x + yP_y + zP_z + 2tP_t - \frac{M}{2}\frac{\partial}{\partial M}$.

one represents algebras in the following way.

case $a = b = c = 0$ This algebra can be written as $(e(3) \oplus (P_t))\Box(D_2)$. In other words, it is generated by the following generators $(P_t, P_x, P_y, P_z, L_x, L_y, L_z, D_2)$.

case $a = b = d = 0$ This algebra can be written as $(e(3) \oplus (P_t))\Box(D_1)$. That is, it is generated by $(P_t, P_x, P_y, P_z, L_x, L_y, L_z, D_1)$.

otherwise The last algebra is of the form $e(3) \oplus (P_t)$. That means that it is generated by $(P_t, P_x, P_y, P_z, L_x, L_y, L_z)$.

In the above description $\oplus$ denotes the simple product of Lie algebras, $\Box$ denotes the semisimple product of Lie algebras, and $e(3)$ denotes the euclidean algebra generated by $(P_x, P_y, P_z)\Box(L_x, L_y, L_z)$ acting on the three dimensional space.

The non-zero commutation relations are :

$[L_x, L_y] = L_z$, $[L_y, L_z] = L_x$, $[L_z, L_x] = L_y$,

$[L_x, P_y] = P_z$, $[L_z, P_z] = -P_y$, $[L_y, P_x] = -Pz$,

$[L_y, P_z] = P_x$, $[L_z, P_x] = P_y$, $[L_z, P_y] = -P_x$,

$[D, P_x] = P_x$, $[D, P_y] = P_y$, $[D, P_z] = P_z$, $[D, P_t] = P_t$,

where D is equal to D_1 or D_2.

Equation (3.1) has not only continuous symmetries but also some discrete ones. For example, transformations $x \to \pm x$, $y \to \pm y$ and $z \to \pm z$ leave Eq. (3.1) invariant. These symmetries were also used to simplify subalgebras.

b) Lists of subalgebras

To obtain a reduction of a PDE to a fewer dimensional PDE one has to obtain equations for surfaces on which the general solution is constant and then map an original PDE onto the quotient structure on which the new equation is defined. The surfaces of the constant value are defined by the system of vector fields. The Frobenius theorem [6] states that the necessary and sufficient condition for a system of vector fields to describe a surface is that they must create a Lie algebra. Therefore, investigation of the possible subalgebras of the Lie algebra describing the symmetry of a PDE is the first step on the way to obtain solutions through the symmetry reduction method.

Because the commutation relations are independent whether D_1 or D_2 is used, the first two algebras are algebraically isomorphic. Therefore, it is sufficient to investigate the algebraic structure of one of them. The corresponding lists of representatives of the conjugacy classes are listed in Table 3.1 and Table 3.2.

Table 3.1: The representatives of conjugacy classes of algebras generated by $(P_x, P_y, P_z, P_t, L_x, L_y, L_z)$; α and β are different from 0; n is the dimension of an algebra; n_1 is the dimension of the corresponding orbit.

Number	Generators	n	n_1	Normalizer
$g_{0,0}$	$L_x, L_y, L_z, P_x, P_y, P_z, P_t$	7	4	$g_{0,0}$
$g_{0,1}$	$L_x, L_y, L_z, P_x, P_y, P_z$	6	3	$g_{0,0}$
$g_{0,2}$	L_x, L_y, L_z, P_t	4	3	$g_{0,2}$
$g_{0,3}$	L_x, L_y, L_z	3	2	$g_{0,2}$
$g_{1,0}$	L_x, P_x, P_y, P_z, P_t	5	4	$g_{1,0}$
$g_{1,1}$	$L_x, P_x + \alpha P_t, P_y, P_z$	4	3	$g_{1,0}$
$g_{1,2}$	L_x, P_x, P_y, P_z	4	3	$g_{1,0}$
$g_{1,3}$	L_x, P_y, P_z, P_t	4	3	$g_{1,0}$
$g_{1,4}$	L_x, P_x, P_t	3	3	$g_{1,4}$
$g_{1,5}$	L_x, P_y, P_z	3	2	$g_{1,0}$
$g_{1,6}$	$L_x, P_x + \alpha P_t$	2	2	$g_{1,4}$
$g_{1,7}$	L_x, P_x	2	2	$g_{1,4}$
$g_{1,8}$	L_x, P_t	2	2	$g_{1,4}$

$g_{1,9}$	L_x	1	1	$g_{1,4}$
$g_{2,0}$	P_x, P_y, P_z, P_t	4	4	$g_{0,0}$
$g_{2,1}$	P_x, P_y, P_z	3	3	$g_{0,0}$
$g_{2,2}$	$P_y, P_z, P_t + \alpha P_x$	3	3	$g_{3,0}$
$g_{2,3}$	P_y, P_z, P_t	3	3	$g_{3,0}$
$g_{2,4}$	P_x, P_y	2	2	$g_{2,0}$
$g_{2,5}$	P_x, P_t	2	2	$g_{2,0}$
$g_{2,6}$	$P_x + \alpha P_t, P_y$	2	2	$g_{5,0}$
$g_{2,7}$	P_x	1	1	$g_{5,0}$
$g_{2,8}$	P_t	1	1	$g_{3,0}$
$g_{2,9}$	$P_x + \alpha P_t$	1	1	$g_{5,0}$
$g_{2,10}$	0	0	0	$g_{0,0}$
$\tilde{g}_{1,10}$	$L_x + \alpha P_t, P_x, P_y, P_z$	4	4	$g_{1,0}$
$\tilde{g}_{1,11}$	$L_x + \alpha P_x, P_y, P_z, P_t$	4	4	$g_{1,0}$
$\tilde{g}_{1,12}$	$L_x + \alpha P_t, P_x + \alpha P_t, P_y, P_z$	4	4	$g_{1,0}$
$\tilde{g}_{1,13}$	$L_x + \alpha P_x + \beta P_t, P_y, P_z$	3	3	$g_{1,0}$
$\tilde{g}_{1,14}$	$L_x + \alpha P_x, P_y, P_z$	3	3	$g_{1,0}$
$\tilde{g}_{1,15}$	$L_x + \alpha P_t, P_y, P_z$	3	3	$g_{1,0}$
$\tilde{g}_{1,16}$	$L_x + \alpha P_t, P_x$	2	2	$g_{1,4}$
$\tilde{g}_{1,17}$	$L_x + \alpha P_x, P_t$	2	2	$g_{1,4}$
$\tilde{g}_{1,18}$	$L_x + \alpha P_t, P_x + \beta P_t$	2	2	$g_{1,4}$
$\tilde{g}_{1,19}$	$L_x + \alpha P_x + \beta P_t$	1	1	$g_{1,4}$
$\tilde{g}_{1,20}$	$L_x + \alpha P_x$	1	1	$g_{1,4}$
$\tilde{g}_{1,21}$	$L_x + \alpha P_t$	1	1	$g_{1,4}$

Table 3.2: The representatives of conjugacy classes of the algebra generated by $(P_x, P_y, P_z, P_t, L_x, L_y, L_z, D)$; α and β are different from 0; n is the dimension of an algebra; n_1 is the dimension of the corresponding orbit.

Number	Generators	n	n_1	Normalizer
$g_{0,0}$	$L_x, L_y, L_z, P_x, P_y, P_z, P_t, D$	8	5	$g_{0,0}$
$g_{0,1}$	$L_x, L_y, L_z, P_x, P_y, P_z, P_t$	7	4	$g_{0,0}$
$g_{0,2}$	$L_x, L_y, L_z, P_x, P_y, P_z, D$	7	4	$g_{0,2}$
$g_{0,3}$	$L_x, L_y, L_z, P_x, P_y, P_z$	6	3	$g_{0,0}$
$g_{0,4}$	L_x, L_y, L_z, P_t, D	5	4	$g_{0,4}$
$g_{0,5}$	L_x, L_y, L_z, P_t	4	3	$g_{0,4}$
$g_{0,6}$	L_x, L_y, L_z, D	4	3	$g_{0,6}$
$g_{0,7}$	L_x, L_y, L_z	3	2	$g_{0,4}$
$g_{1,0}$	$L_x, P_x, P_y, P_z, P_t, D$	6	5	$g_{1,0}$
$g_{1,1}$	L_x, P_x, P_y, P_z, P_t	5	4	$g_{1,0}$
$g_{1,2}$	L_x, P_x, P_y, P_z, D	5	4	$g_{1,2}$
$g_{1,3}$	L_x, P_y, P_z, P_t, D	5	4	$g_{1,3}$
$g_{1,4}$	L_x, P_x, P_y, P_z	4	3	$g_{1,0}$
$g_{1,5}$	L_x, P_y, P_z, P_t	4	3	$g_{1,0}$
$g_{1,6}$	$L_x, P_x + \alpha P_t, P_y, P_z$	4	3	$g_{1,1}$
$g_{1,7}$	L_x, P_x, P_t, D	4	4	$g_{1,7}$
$g_{1,8}$	L_x, P_y, P_z, D	4	3	$g_{1,8}$
$g_{1,9}$	L_x, P_x, P_t	3	3	$g_{1,7}$
$g_{1,10}$	L_x, P_y, P_z	3	2	$g_{1,0}$
$g_{1,11}$	L_x, P_x, D	3	3	$g_{1,11}$
$g_{1,12}$	L_x, P_t, D	3	3	$g_{1,12}$

$g_{1,13}$	L_x, P_x	2	2	$g_{1,7}$
$g_{1,14}$	L_x, P_t	2	2	$g_{1,7}$
$g_{1,15}$	$L_x, P_x + \alpha P_t$	2	2	$g_{1,9}$
$g_{1,16}$	L_x, D	2	2	$g_{1,16}$
$g_{1,17}$	L_x	1	1	$g_{1,7}$
$g_{2,0}$	P_x, P_y, P_z, P_t, D	5	5	$g_{0,0}$
$g_{2,1}$	P_x, P_y, P_z, P_t	4	4	$g_{0,0}$
$g_{2,2}$	P_x, P_y, P_z, D	4	4	$g_{0,2}$
$g_{2,3}$	P_y, P_z, P_t, D	4	4	$g_{1,3}$
$g_{2,4}$	P_x, P_y, P_z	3	3	$g_{0,0}$
$g_{2,5}$	$P_x + \alpha P_t, P_y, P_z,$	3	3	$g_{1,6}$
$g_{2,6}$	P_y, P_z, P_t	3	3	$g_{1,0}$
$g_{2,7}$	P_x, P_t, D	3	3	$g_{1,7}$
$g_{2,8}$	P_y, P_z, D	3	3	$g_{1,8}$
$g_{2,9}$	P_x, P_y	2	2	$g_{2,0}$
$g_{2,10}$	P_x, P_t	2	2	$g_{1,0}$
$g_{2,11}$	$P_x + \alpha P_t, P_y$	2	2	$g_{2,1}$
$g_{2,12}$	P_x, D	2	2	$g_{1,11}$
$g_{2,13}$	P_t, D	2	2	$g_{0,4}$
$g_{2,14}$	P_x	1	1	$g_{1,0}$
$g_{2,15}$	P_t	1	1	$g_{0,0}$
$g_{2,16}$	$P_x + \alpha P_t$	1	1	$g_{1,1}$
$g_{2,17}$	D	1	1	$g_{0,6}$
$g_{2,18}$	0	0	0	$g_{0,0}$
$\tilde{g}_{1,18}$	$L_x + \alpha D, P_x, P_y, P_z, P_t$	5	5	$g_{1,0}$
$\tilde{g}_{1,19}$	$L_x + \alpha D, P_x, P_y, P_z$	4	4	$g_{1,2}$

$\tilde{g}_{1,20}$	$L_x + \alpha D, P_y, P_z, P_t$	4	4	$g_{1,3}$
$\tilde{g}_{1,21}$	$L_x + \alpha P_t, P_x, P_y, P_z$	4	4	$g_{1,1}$
$\tilde{g}_{1,22}$	$L_x + \alpha P_t, P_x + \beta P_t, P_y, P_z$	4	4	$g_{1,1}$
$\tilde{g}_{1,23}$	$L_x + \alpha P_x, P_y, P_z, P_t$	4	4	$g_{1,1}$
$\tilde{g}_{1,24}$	$L_x + \alpha D, P_x, P_t$	3	3	$g_{1,7}$
$\tilde{g}_{1,25}$	$L_x + \alpha D, P_y, P_z$	3	3	$g_{1,8}$
$\tilde{g}_{1,26}$	$L_x + \alpha P_x, P_y, P_z$	3	3	$g_{1,1}$
$\tilde{g}_{1,27}$	$L_x + \alpha P_t, P_y, P_z$	3	3	$g_{1,1}$
$\tilde{g}_{1,28}$	$L_x + \alpha P_x + \beta P_t, P_y, P_z$	3	3	$g_{1,1}$
$\tilde{g}_{1,29}$	$L_x + \alpha D, P_x$	2	2	$g_{1,11}$
$\tilde{g}_{1,30}$	$L_x + \alpha D, P_t$	2	2	$g_{1,12}$
$\tilde{g}_{1,31}$	$L_x + \alpha P_x, P_t$	2	2	$g_{1,9}$
$\tilde{g}_{1,32}$	$L_x + \alpha P_t, P_x$	2	2	$g_{1,9}$
$\tilde{g}_{1,33}$	$L_x + \alpha P_z, P_x + \beta P_t$	2	2	$g_{1,9}$
$\tilde{g}_{1,34}$	$L_x + \alpha P_x + \beta P_t$	1	1	$g_{1,9}$
$\tilde{g}_{1,35}$	$L_x + \alpha P_x$	1	1	$g_{1,9}$
$\tilde{g}_{1,36}$	$L_x + \alpha P_t$	1	1	$g_{1,9}$
$\tilde{g}_{1,37}$	$L_x + \alpha D$	1	1	$g_{1,16}$

Each entry in Tables 3.1 - 3.2 consists of a subalgebra specified by its generators then the dimension of the given subalgebra, the generic dimension of the orbit of the associated subgroup and finally its normalizer in the whole algebra.

The classes of subalgebras that have 4-dimensional orbits in the space of the independent variables give reductions to an algebraic equation. Classes that have 3-dimensional orbits in the space of the independent variables give reductions to ordinary differential equations.

c) *Geometry of symmetry variables*

There is a certain amount of arbitrariness in the choice of the symmetry variables. The symmetry variables are constructed as integrals of the vector fields. But any smooth function of the symmetry variables is also an integral curve of these vector fields, therefore the choice is not unique. Whenever it was possible I have tried to choose the symmetry variables in such a way so that they had a possible simple geometric interpretation.

The symmetry variable ξ of a subalgebra that does not contain dilation as one of its generators leads to a reduced equation with solutions in the form $M(x,y,z,t) = f(\xi)$ where ξ is a function of x,y,z,t. If dilation belongs to the set of generators then generally speaking the reduction is of the form $M(x,y,z,t) = \rho f(\xi)$. In the last expression ρ is also a function of independent variables x,y,z,t. The reductions to ordinary differential equations of the partial differential equation (3.1) are listed in Tables 3.3 - 3.5.

It is useful to discuss the action of subgroups in terms of geometry of its orbits. The obtained information allows one to predict what kind of symmetries might be observable in the system. Solutions which are stable or at least metastable having a relatively long life time should be observed in numerical or actual physical experiments, when their energy is a local minimum of the energy function in the space of states.

The simplest possible geometry is given by the spatially homogeneous (constant) solutions. They might be time-independent or time-dependent. The value of the order parameter for time-independent spatially-homogeneous solutions is given by solutions to the algebraic equation

$$a + bf + cf^3 + df^5 = 0 \qquad (3.2)$$

These solutions recover the results of the standard mean field (without spatial in-

Table 3.3: Reductions of the equation $\partial_t\eta + \partial_{xx}\eta + \partial_{yy}\eta + \partial_{zz}\eta = a + b\eta + c\eta^3 + d\eta^5$ to ordinary differential equations. At least two of the coefficients a, b, c, d one of which must be c and/or d are different form zero; $\eta(x, y, z, t) = f(\xi)$ where ξ is a function of x, y, z, t; α is different from 0.

No	ξ	Reduced Equation	Painlevé Test
1	t	$\frac{df}{d\xi} = a + bf + cf^3 + df^5$	1^{st}-order
2	x	$\frac{d^2f}{d\xi^2} = a + bf + cf^3 + df^5$	for $a = 0$ or $d = 0$
3	$x + \alpha t$	$\frac{d^2f}{d\xi^2} + \alpha\frac{df}{d\xi} = a + bf + cf^3 + df^5$	for $[a = c = 0$ and $b = -\frac{3\alpha^2}{16}]$
			or $[a = d = 0$ and $b = -\frac{2\alpha^2}{9}]$
4	$(x^2 + y^2)^{1/2}$	$\frac{d^2f}{d\xi^2} + \frac{1}{\xi}\frac{df}{d\xi} = a + bf + cf^3 + df^5$	no
5	$(x^2 + y^2 + z^2)^{1/2}$	$\frac{d^2f}{d\xi^2} + \frac{2}{\xi}\frac{df}{d\xi} = a + bf + cf^3 + df^5$	no

Table 3.4: Reductions of the equation $\partial_t\eta + \partial_{xx}\eta + \partial_{yy}\eta + \partial_{zz}\eta = c\eta^3$ to ordinary differential equations. The coefficient c and the constant α are different from 0; $\eta(x,y,z,t) = \rho f(\xi)$ where ξ and ρ are functions of x, y, z, t.

No	ξ	ρ	Reduced Equation	Painlevé Test
1	t	1	$\frac{df}{d\xi} = cf^3$	1st-order
2	x	1	$\frac{d^2f}{d\xi^2} = cf^3$	yes
3	$x + \alpha t$	1	$\frac{d^2f}{d\xi^2} + \alpha\frac{df}{d\xi} = cf^3$	no
4	$(x^2+y^2)^{1/2}$	1	$\frac{d^2f}{d\xi^2} + \frac{1}{\xi}\frac{df}{d\xi} = cf^3$	no
5	$(x^2+y^2+z^2)^{1/2}$	1	$\frac{d^2f}{d\xi^2} + \frac{2}{\xi}\frac{df}{d\xi} = cf^3$	no
6	$\tan^{-1}(\frac{y}{x})$	$(x^2+y^2)^\alpha$	$\frac{d^2f}{d\xi^2} + f = cf^3$	yes
7	$-\frac{2\alpha}{\alpha^2+1}[\tan^{-1}(\frac{y}{x}) - \frac{1}{2}\alpha\log(x^2+y^2)]$	$(\frac{4\alpha^2}{(\alpha^2+1)(x^2+y^2)})^{1/2}$	$\frac{d^2f}{d\xi^2} + \frac{df}{d\xi} + \frac{\alpha^2+1}{4\alpha^2}f = cf^3$	for $\alpha = \pm 3i$
8	$\cot^{-1}(\frac{z}{\sqrt{(x^2+y^2)}})$	$(x^2+y^2+z^2)^{-1/2}$	$\frac{d^2f}{d\xi^2} + \cot(\xi)\frac{df}{d\xi} = cf^3$	no
9	$\frac{t}{x^2}$	$\frac{1}{x}$	$\xi^2\frac{d^2f}{d\xi^2} + (5\xi^2 + \frac{1}{2})\frac{df}{d\xi} + 2f = cf^3$	no
10	$\frac{t}{2(y^2+z^2)}$	$(y^2+z^2)^{-1/2}$	$\xi^2\frac{d^2f}{d\xi^2} + (4\xi + \frac{1}{2})\frac{df}{d\xi} + f = cf^3$	no
11	$\frac{t}{x^2+y^2+z^2}$	$(x^2+y^2+z^2)^{-1/2}$	$\xi^2\frac{d^2f}{d\xi^2} + (6\xi + 1)\frac{df}{d\xi} = cf^3$	no

Table 3.5: Reductions of the equation $\partial_t \eta + \partial_{xx}\eta + \partial_{yy}\eta + \partial_{zz}\eta = d\eta^5$ to ordinary differential equations. The coefficient d and the constant α are different from 0; $\eta(x,y,z,t) = \rho f(\xi)$ where ξ and ρ are functions of x, y, z, t.

No	ξ	ρ	Reduced Equation	Painlevé Test
1	t	1	$\frac{df}{d\xi} = df^5$	1^{st}-order
2	x	1	$\frac{d^2f}{d\xi^2} = df^5$	yes
3	$x + \alpha t$	1	$\frac{d^2f}{d\xi^2} + \alpha\frac{df}{d\xi} = df^5$	no
4	$(x^2+y^2)^{1/2}$	1	$\frac{d^2f}{d\xi^2} + \frac{1}{\xi}\frac{df}{d\xi} = df^5$	no
5	$(x^2+y^2+z^2)^{1/2}$	1	$\frac{d^2f}{d\xi^2} + \frac{2}{\xi}\frac{df}{d\xi} = df^5$	yes
6	$\frac{1}{2}\tan^{-1}(\frac{y}{x})$	$(x^2+y^2)^{\alpha}$	$\frac{d^2f}{d\xi^2} + f = df^5$	yes
7	$-\frac{2\alpha}{\alpha^2+1}[\tan^{-1}(\frac{y}{x}) - \frac{1}{2}\alpha\log(x^2+y^2)]$	$(\frac{4\alpha^2}{(\alpha^2+1)(x^2+y^2)})^{1/4}$	$\frac{d^2f}{d\xi^2} + \frac{df}{d\xi} + \frac{\alpha^2+1}{4\alpha^2}f = df^5$	for $\alpha = \pm 2i$
8	$\cot^{-1}(\frac{z}{\sqrt{(x^2+y^2)}})$	$(x^2+y^2+z^2)^{-1/4}$	$\frac{d^2f}{d\xi^2} + \cot(\xi)\frac{df}{d\xi} + \frac{1}{4}f = df^5$	no
9	$\frac{t}{x^2}$	$\frac{1}{\sqrt{x}}$	$\xi^2\frac{d^2f}{d\xi^2} + (4\xi^2 + \frac{1}{2})\frac{df}{d\xi} + \frac{3}{4}f = df^5$	no
10	$\frac{t}{2(y^2+z^2)}$	$(y^2+z^2)^{-1/4}$	$\xi^2\frac{d^2f}{d\xi^2} + (3\xi + \frac{1}{2})\frac{df}{d\xi} + \frac{1}{4}f = df^5$	no
11	$\frac{t}{x^2+y^2+z^2}$	$(x^2+y^2+z^2)^{-1/4}$	$\xi^2\frac{d^2f}{d\xi^2} + (6\xi + 1)\frac{df}{d\xi} - f = df^5$	no

homogeneities) Landau theory.

The next class of solutions is related to stationary or travelling plane waves. In the most general case 3 there are only two other symmetry variables left. One of them describes cylindrical patterns and the other one spherical patterns.

For the case where only $c \neq 0$ or only $d \neq 0$ there are a few more symmetry variables. The time-independent solutions are constant on half-planes enumerated by the angle, on logarithmic spirals, and on cones. The time-dependent solutions include solutions constant on moving planes, on expanding cylinders, and on expanding spheres. In the case $c \neq 0$ the relation between ξ and ρ is $\rho = \sqrt{\xi/t}$ while in the case $d \neq 0$ this relation is $\rho = (\xi/t)^{1/4}$. The surfaces of the constant values for some of these surfaces are sketched in Fig. 3.1.

d) Solutions of algebraic and first order equations from Tables 3.3 - 3.5

Fortunately, it is possible to solve analytically at least some of the listed ODE's. Those for which solutions are known include the only first-order equation and all the equations that pass the Painlevé Test. The first-order equation is of the following form.

$$\frac{df}{d\xi} = a + bf + cf^3 + df^5 \tag{3.3}$$

Such an equation is integrable using the fractional parts method. The knowledge of the roots of the quintic polynomial on the right hand side is necessary to apply this method. There are no analytic expressions to solve such an equation but if at least one root is found numerically the equation can be reduced to a quartic one and the rest of the procedure is analytical. The symmetry variable that leads to this particular reduction is $\xi = t$. Solutions then are time-dependent and spatially homogenous. If at the given time $t = t_0$ the initial value $f(t_0) = f_0$ then the behaviour of the solution $f(t)$ for $t_0 < t$ is determined by the position of f_0 relative

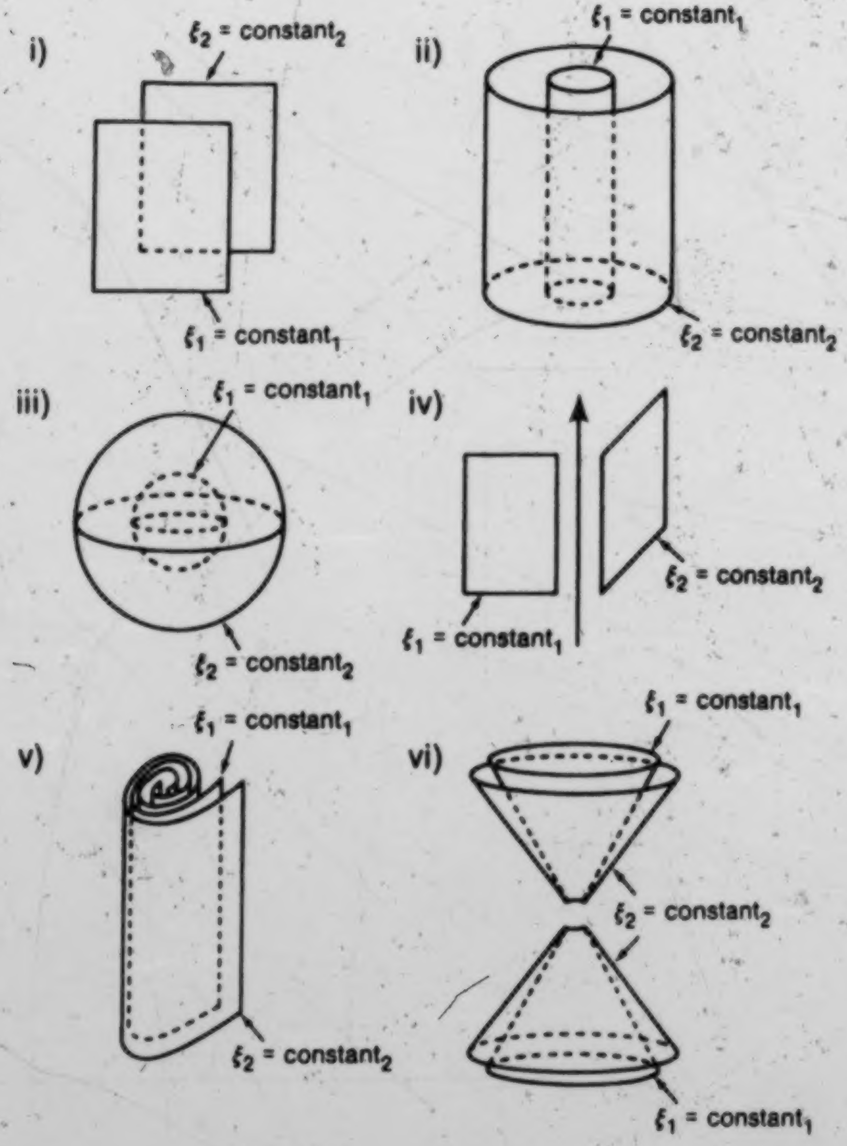
i)
ξ_2 = constant$_2$
ξ_1 = constant$_1$
ii)
ξ_1 = constant$_1$
ξ_2 = constant$_2$
iii)
ξ_1 = constant$_1$
ξ_2 = constant$_2$
iv)
ξ_2 = constant$_2$
ξ_1 = constant$_1$
v)
ξ_1 = constant$_1$
ξ_2 = constant$_2$
vi)
ξ_1 = constant$_1$
ξ_2 = constant$_2$
ξ_1 = constant$_1$

Figure 3.1: Geometries of some symmetry variables obtained by reduction of the Eq. $\partial_t M + \partial_{xx} M + \partial_{yy} M + \partial_{zz} M = \gamma M^q$ for $q = 3, 5$. i) planar waves; ii) cylindrical waves; iii) spherical waves; iv) planes indexed by angle; v) spiral sheets; vi) conical symmetry (each surface consists of two sheets).

to other roots of the polynomial (3.2). If f_0 is outside of the range of roots then $f(t)$ tends to $\pm\infty$ as t tends to ∞. Otherwise $f(t)$ tends to one of the stable stationary solutions (if such exist). This behaviour is illustrated in Fig. 3.2. There is at least one unstable solution for the coefficient $d > 0$ and at least one stable solution for $d < 0$. There may be up to three unstable or stable solutions for this reduction. Here the word stability means stability of a solution of the reduced ODE (3.3).

e) Equations satisfying the Painlevé test

The calculations of the Painlevé test were performed using a program written in the symbolic language $MACSYMA^{TM}$ by D.Rand [54]. The above calculations were independently checked using program also written in $MACSYMA^{TM}$ by W.Hereman [24].

The following equations from Tables 3.3 - 3.5 passed the Painlevé test:

$$\frac{d^2f}{d\xi^2} = a + bf + cf^3 \tag{3.4}$$

$$\frac{d^2f}{d\xi^2} = bf + cf^3 + df^5 \tag{3.5}$$

$$\frac{d^2f}{d\xi^2} + \frac{1}{v}\frac{df}{d\xi} = a + bf + cf^3 + df^5 \quad for \tag{3.6}$$

$$a = d = 0 \ \ and \ \ b = -\frac{2v^2}{9} \quad or \tag{3.7}$$

$$a = c = 0 \ \ and \ \ b = -\frac{3v^2}{16} \tag{3.8}$$

$$\frac{d^2f}{d\xi^2} + \frac{2}{\xi}\frac{df}{d\xi} = df^5 \tag{3.9}$$

$$\frac{d^2f}{d\xi^2} + f = cf^3 \tag{3.10}$$

$$\frac{d^2f}{d\xi^2} + f = df^5 \tag{3.11}$$

$$\frac{d^2f}{d\xi^2} + \frac{df}{d\xi} + \frac{\alpha^2+1}{4\alpha^2}f = cf^3 \quad for \ \ \alpha = \pm 3i \tag{3.12}$$

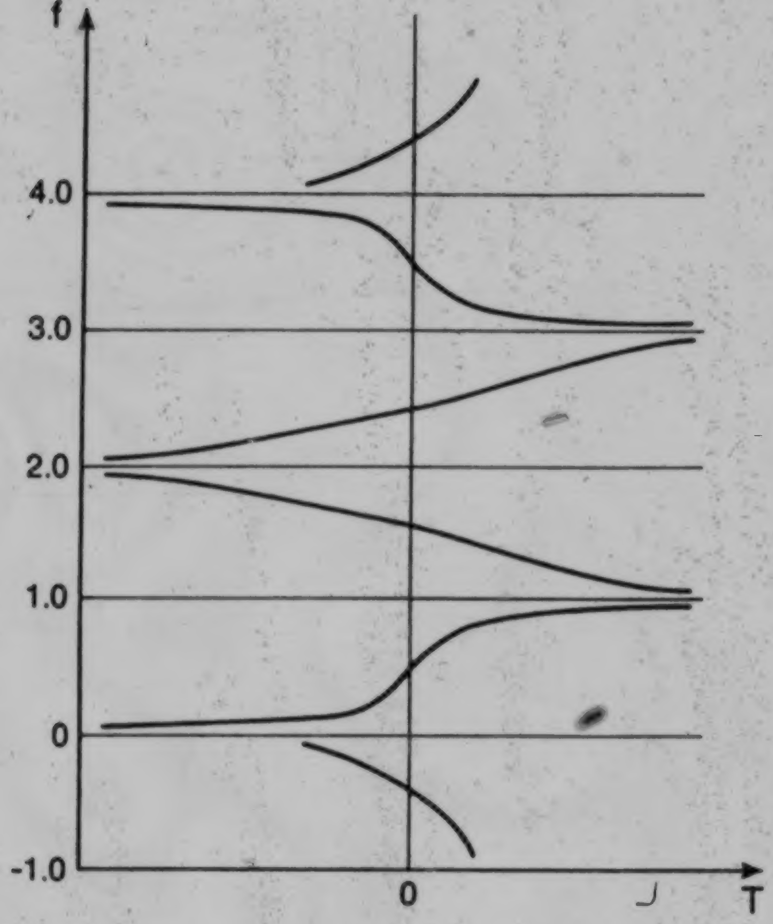

Figure 3.2: Solutions of the equation $f'(t) = f(f-1)(f-2)(f-3)(f-4)$ for different values of the initial value f_0. The singular solutions at 1.0 and 3.0 are stable while those at 0.0, 2.0, and 4.0 are not.

$$\frac{d^2f}{d\xi^2} + \frac{df}{d\xi} + \frac{\alpha^2+1}{4\alpha^2} f = df^3 \qquad \mathit{for} \quad \alpha = \pm 2i \tag{3.13}$$

Solutions of equation (3.4)

Equations (3.4) and (3.5) appear many times in different contexts in this thesis. Because of this they deserve more attention. The solutions to equation (3.4) in the case when $a = 0$ were discussed in [66] while in the general case in [22].

The first observation is that the second order derivative is a smooth function of its dependent variable. That means that a unique solution exists for any initial conditions. Both equations have singular solutions. They are given by the roots of the polynomial and were already discussed as solutions of Eq. (3.2).

There are many nonsingular solutions to Eq. (3.4) that can be written in a closed form as well. Multiplication of Eq. (3.4) by $df/d\xi \neq 0$ (without any loss of generality because singular solutions were already discussed) followed by integration gives the following equation

$$\frac{1}{2}(\frac{df}{d\xi})^2 = K + af + \frac{b}{2}f^2 + \frac{c}{4}f^4. \tag{3.14}$$

It is obvious that real solutions exist only when for the initial point (ξ_0, f_0) the right hand polynomial is positive and that in the opposite case solutions are purely imaginary. Additionally Eq. (3.14) is symmetric with respect to the reflection $x \rightarrow -x$. After change of the role of the independent and dependent variables, Eq. (3.14) is integrable in terms of Jacobi elliptic functions. The required integrals are listed in the handbook by Byrd and Friedmann [10]. The Jacobian elliptic functions are meromorphic periodic functions defined on the complex plane. Because physically interesting variables are real, only the real sections of these functions are interesting. Some of the solutions have periodic poles on the real axis. It seems that in the application to the magnetic systems such solutions should be rejected. There is, however, a certain degree of physical interest in functions representing such behaviour in combustion processes [4].

The solutions of Eq. (3.14) contain $sn, cn, dn, \ldots$ Jacobi elliptic functions. The full list of solutions was presented by A.M.Grundland [22]. Unfortunately, the size of this list prohibits from copying it into this thesis. For every position of the initial condition $f(x_0)$ with respect to the roots of the polynomial standing on the right hand side of Eq. (3.14) there is a specific solution. The sketch illustrating generic regions in which different types of solutions appear is presented in Fig. 3.3.

Solutions of equation (3.5)

The solutions to Eq. (3.5) were discussed earlier by Winternitz et al [67]. To solve this equation the first integration proceeds in the same way as in the case of Eq. (3.4). The next step is a substitution $f = \pm\sqrt{h}$. After these two operations the following equation is obtained

$$\frac{1}{8}\left(\frac{dh}{d\xi}\right)^2 = K + \frac{b}{2}h^2 + \frac{c}{4}h^3 + \frac{d}{6}h^4 \tag{3.15}$$

In its essence this equation is very similar to Eq. (3.14). The solutions to Eq. (3.15) are also obtained by the inversion of the roles of the independent and dependent variables. They are expressed in terms of the Jacobi elliptic functions. The problem of a proper continuation of a solution of Eq. (3.5) arises when the corresponding solution of Eq. (3.15) passes through zero. The continuity of the first derivative allows one to solve this problem.

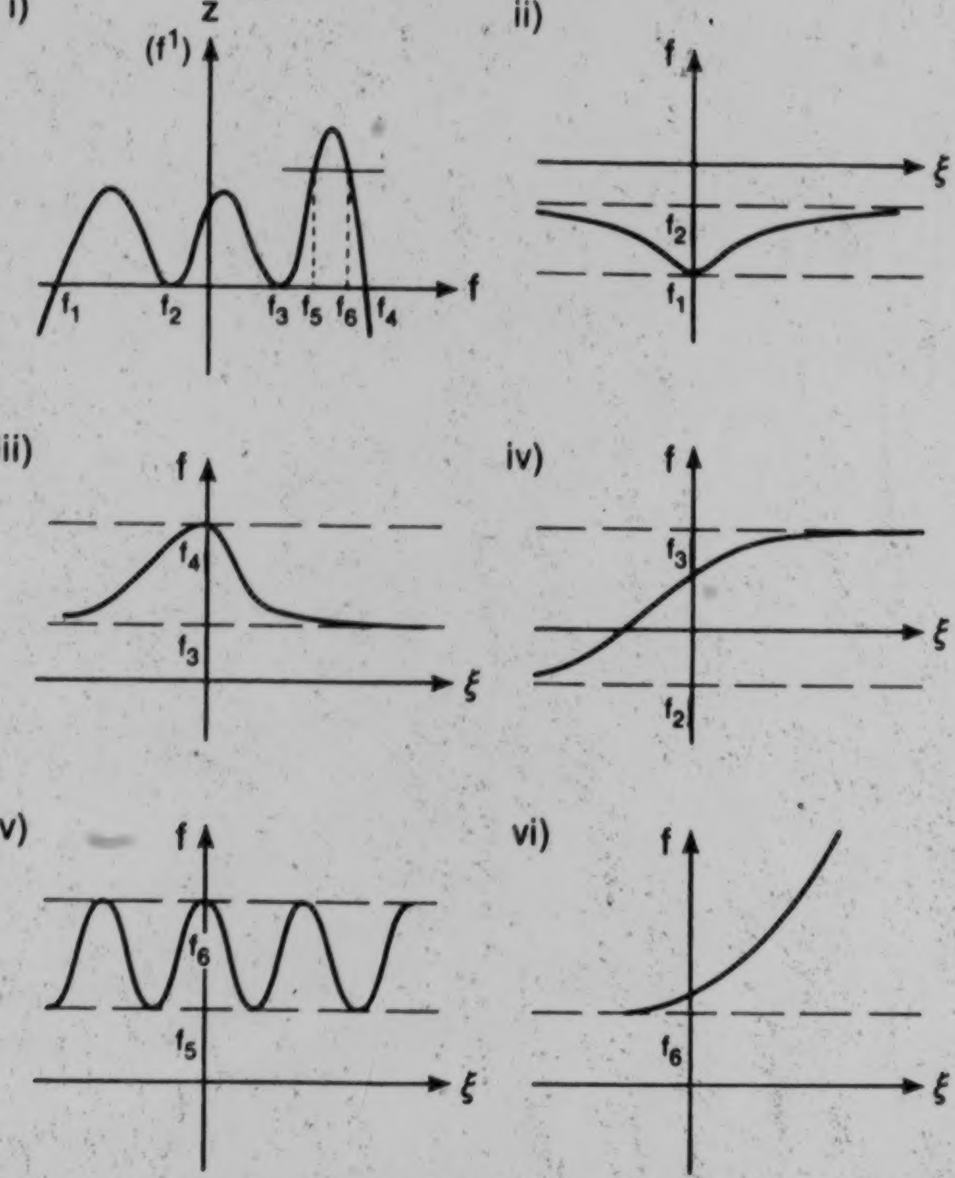
i)
z
(f1)
f
f1
f2
f3
f5
f6
f4
ii)
f
ξ
f2
f1
iii)
f
f4
f3
ξ
iv)
f
f3
ξ
f2
v)
f
f6
f5
ξ
vi)
f
f6
ξ

Figure 3.3: Some real solutions of equation $f''(x) = a + bf + cf^3 + df^5$ for different values of the initial value f_0. i) f' as a function of f; ii) bump; iii) antibump; iv) kink; v) periodic; vi) divergent.

Solutions of equation (3.6) with condition (3.7)

Using substitutions $f = \beta g$ and $\zeta = \xi / v$ Eq. (3.6) with condition (3.7) can be rewritten in a form where there are no free coefficients left, i.e.

$$\frac{d^2 g}{d\zeta^2} + \frac{dg}{d\zeta} = -\frac{2}{9} g + 2g^3 \tag{3.16}$$

where $\beta = v\sqrt{2/c}$. In a similar way Eq. (3.8) is transformed to

$$\frac{d^2 g}{d\zeta^2} + \frac{dg}{d\zeta} = -\frac{3}{16} g + \frac{3}{4} g^5 \tag{3.17}$$

where $\beta = (\frac{3v^2}{4d})^{1/4}$. The general solution of Eq. (3.16) can be found in [28, page 548]

$$f(\zeta) = -\frac{K_1 i}{3} \exp(-\frac{\zeta}{3})\ sd(\sqrt{2}(K_1 \exp(-\frac{\zeta}{3} + K_2), 2^{-1/2})$$

Therefore, for $c > 0$ most of the solutions are purely imaginary while for $c < 0$ one should expect many real solutions.

Solutions of equation (3.6) with condition (3.8)

The general solution of equation (3.17) was obtained by P.Winternitz et al [67]. After substitution back to the old variables the solution of Eq. (3.6) with condition (3.8) is in the form

$$f(\xi) = [-(\frac{3}{8d})^{1/2} \exp -(\frac{\xi}{2}) W[\exp -(\frac{\xi}{2})]]^{1/2}$$

where W is an arbitrary solution of the equation

$$\dot{W}^2 = W^4 + CW \tag{3.18}$$

which has solutions expressed by Jacobi elliptic functions.

It should be noted that for the solution f to be real the product $\sqrt{c}W$ must be real. It means that for negative values of the coefficient c one is interested in purely imaginary solutions of Eq. (3.18).

Solutions of equation (3.9)

An analytic solution of the Eq. (3.9) was found by Winternitz et al [67]. Solutions of this equation as well as of some its generalizations are also discussed by J.Dixon et al [13].

The substitution $f = \pm\sqrt{h}$ into (3.9) gives the equation

$$\frac{d^2h}{d\xi^2} = \frac{1}{2h}(\frac{dh}{d\xi})^2 - \frac{2}{\xi}\frac{dh}{d\xi} + 2dh^3. \tag{3.19}$$

The next transformation is of the form $h(\xi) = \lambda(\xi)W(\zeta)$ where ζ is a function of the variable ξ. After such substitution into Eq. (3.19) functions ζ, W, λ can be chosen in such a way that Eq. (3.19) is in the canonical Painlevé form number XXX. The last substitution is relatively general and can be encountered in many reductions to the Painlevé types.

In the specific case of Eq. (3.19) the function λ can be chosen as $\lambda(\xi) = \sqrt{3/4d}\xi^{-1}$, the function η is chosen as $\eta(\xi) = \ln(\xi)$. Such a transformed equation is in the form

$$\frac{d^2W}{d\eta^2} = \frac{1}{2W}(\frac{dW}{d\eta})^2 + \frac{3}{2}W^3 + \frac{1}{2}W \tag{3.20}$$

But this equation can be once integrated to the form

$$(\frac{dW}{d\eta})^2 = W^4 + 2W^2 + 4rW \tag{3.21}$$

where r is an integration constant. The last equation is an equation for Jacobi elliptic functions (3.15).

Going back to the original variables f and ξ the formula

$$f(\xi) = \sqrt{\frac{3W(\ln(\xi))}{4d\xi}} \tag{3.22}$$

is obtained. To obtain real solutions the function W must be real and must have the same sign as the coefficient d. It is possible to discuss solutions of Eq. (3.21) in more detail. For the function W to be real the integration constant r must be real as well. In such a case the polynomial which is on the right hand side of Eq. (3.21) has only one or two real roots, namely when $r = 0$ then 0 is a double root and when $r \neq 0$ then 0 is a single root and there is one more real root. For an arbitrary choice of the integration constant r real solutions are divergent.

Solutions of equation (3.10)

The solutions of Eq. (3.10) are obtained through the same procedure as those of Eq. (3.4). The singular solutions are $f = 0$ for $c < 0$ and $f = 0, \pm c^{-1/2}$ for $c > 0$. The change of the dependent variable to $f = g \mid c \mid^{-1/2}$, multiplication by $df/d\xi$ and integration gives the following equation

$$(\frac{df}{d\xi})^2 = \frac{\epsilon}{2} f^4 - f^2 - \frac{r}{2} \tag{3.23}$$

where $\epsilon = sign(c)$. Because of the nature of the variable ξ which corresponds to the angle in the cylindrical system of coordinates it is important to discuss the periodic solutions with the period equal to $2\pi/n$ where n is an integer.

Let $\epsilon = 1$ then there are five possible cases. The first one when $r > 0$ is characterized by the fact that all solutions have singularities. Solutions are

$$g(\xi) = \sqrt{1 + \sqrt{1 + r}}\ nc(\sqrt{2}(1 + r)^{1/4}(\xi - \xi_0), k)$$

where $k = \sqrt{\frac{1}{2}(1 - (1 + r^{-1/2})}$. In the second case $r = 0$ the solution is

$$g(\xi) = \frac{\sqrt{2}}{\cos(\sqrt{2}(\xi - \xi_0))}.$$

The third case when $-1 < r < 0$ seems to be the most interesting. Besides divergent solutions of the form

$$g(\xi) = \sqrt{1 + \sqrt{1 + r}}\ ns(\sqrt{1 + \sqrt{1 + r}}(\xi - \xi_0), k)$$

where

$$k = \sqrt{\frac{1-\sqrt{1+r}}{1+\sqrt{1+r}}}$$

there is only one periodic solution

$$g(\xi) = \sqrt{1-\sqrt{1+r_1}}\; sn(\sqrt{1+\sqrt{1+r_1}}(\xi-\xi_0), k_1)$$

where

$$k_1 = \sqrt{\frac{1-\sqrt{1+r_1}}{1+\sqrt{1+r_1}}}.$$

The period of this solution corresponds to the only solutions of the transcendental equation

$$\frac{\pi}{2}\sqrt{1+\sqrt{1+r_1}} = K(k_1)$$

where K is the first complete elliptic function. The integration constant r_1 has a subscript 1 to distinguish it from integration constants appearing in the case $\epsilon = -1$.

In the case $r = -1$ the solution is of the form

$$g(\xi) = -\frac{\exp(2\sqrt{2}(\xi-\xi_0) - K}{\exp(2\sqrt{2}(\xi-\xi_0) + K}$$

It has singularities when the denominator is equal to 0.

The last remaining case $r < -1$ has a solution:

$$g(\xi) = \pm\sqrt{\sqrt{-r}\frac{1+cn((-r)^{1/4}(\xi-\xi_0), k)}{1-cn((-r)^{1/4}(\xi-\xi_0), k)}}$$

where

$$k = \sqrt{\frac{1}{2}(1+(-r)^{-1/2})}$$

These solutions have singularities for $\xi = \xi_0 + 2K(k)$.

Equation (3.23) with $\epsilon = -1$ has more solutions without divergencies. The singular solutions are $f = 0$ and $f = \pm i$. Real solutions exist for the integration constant $r > 0$. There is a countable number of them and they can be written as:

$$g(\xi) = \pm\sqrt{\sqrt{1+r_n}-1}\; cn(\sqrt{2}(1+r_n)^{1/4}(\xi-\xi_0), k_n)$$

where $n \geq 2$

$$k_n = \sqrt{\frac{1}{2}(1 - (1 + r_n)^{-1/2}}$$

and r_n is a solution of the transcendental equation

$$K(k_n) = \frac{\pi}{\sqrt{2}n}(1 + r_n)^{1/4}.$$

Solutions of equation (3.11)

As equation (3.10) is a special case of Eq. (3.4), Eq. (3.11) is a special case of (3.5). Therefore, its solutions are also given in terms of the Jacobi elliptic functions. Due to the special character of the symmetry variable one distinguishes here a countable family of continuous solutions. However, the number of possible cases is so big here that explicit formulas are omitted here.

Solutions of equation (3.12) and equation (3.13)

Equation (3.12) for $\alpha = \pm 3i$ as well as equation (3.13) for $\alpha = \pm 2i$ were already discussed in this chapter. Their solutions were listed in the process of discussion of solutions of Eqs. (3.16,3.17).

3.2 The Landau-Ginzburg equation with mixed signature of derivatives

This chapter contains the results of the application of the symmetry reduction method to Eq. (1.27) with $\epsilon = -1$

$$\partial_t M + \frac{\partial^2 M}{\partial x^2} - \frac{\partial^2 M}{\partial y^2} - \frac{\partial^2 M}{\partial z^2} = a + bM + cM^3 + dM^5. \tag{3.24}$$

The basic plan of this chapter is the same as the previous one. Namely, the following points are discussed:

1. Algebras
2. System of subalgebras
3. The geometry of ODE's
4. Algebraic and first order equations
5. The results of the Painlevé Test and solutions of some ODE's

a) The Lie algebras of the Landau-Ginzburg equation with mixed signature

As in the previous case of Eq. (3.1) we have three possible symmetry groups. They are quite similar with the exception of the fact that two of the rotations L_y, L_z are substituted by the hyperbolic rotations K_y, K_z, where

$$K_y = x\partial_z + z\partial_x \qquad K_z = y\partial_x + x\partial_y$$

The new nonzero commutation relations are:
$[L_x, K_y] = -K_z$, $[L_x, K_z] = K_y$, $[K_y, K_z] = L_x$,
$[K_y, P_x] = -P_z$, $[K_y, P_z] = -P_x$, $[K_z, P_x] = -P_y$, $[K_z, P_y] = -P_x$.

The symmetry structure for Eq. (3.24) is much richer than the corresponding structure for Eq. (3.1) because the algebra generated by (L_x, K_y, K_z) has a much more complicated system of subalgebras then the one generated by (L_x, L_y, L_z). There are some discrete symmetries in this case and they were also used to simplify subalgebras.

b) Systems of subalgebras

The algebraic structure of an algebra that does not contain the scaling operator D_i is presented in Table 3.6.

Table 3.6: The representatives of conjugacy classes of the algebra generated by $(P_x, P_y, P_z, P_t, L_x, K_y, K_z)$; α and β are different from 0; n is the dimension of an algebra; n_1 is the dimension of the corresponding orbit.

Number	Generators	n	n_1	Normalizer
$g_{0,0}$	$L_x, K_y, K_z, P_x, P_y, P_z, P_t$	7	4	$g_{0,0}$
$g_{0,1}$	$L_x, K_y, K_z, P_x, P_y, P_z$	6	3	$g_{0,0}$
$g_{0,2}$	L_x, K_y, K_z, P_t	4	3	$g_{0,2}$
$g_{0,3}$	L_x, K_y, K_z	3	2	$g_{0,2}$
$g_{1,0}$	$L_x + K_y, K_z, P_x, P_y, P_z, P_t$	6	4	$g_{1,0}$
$g_{1,1}$	$L_x + K_y, K_z, P_x, P_y, P_z$	5	3	$g_{1,0}$
$g_{1,2}$	$L_x + K_y, K_z, P_x + P_y, P_z, P_t$	5	4	$g_{1,2}$
$g_{1,3}$	$L_x + K_y, K_z, P_x + P_y, P_t + \alpha P_z$	4	4	$g_{1,2}$
$g_{1,4}$	$L_x + K_y, K_z, P_x + P_y, P_t$	4	4	$g_{1,2}$
$g_{1,5}$	$L_x + K_y, K_z, P_x + P_y, P_z$	4	3	$g_{1,2}$
$g_{1,6}$	$L_x + K_y, K_z, P_x + P_y$	3	3	$g_{1,2}$
$g_{1,7}$	$L_x + K_y, K_z, P_t$	3	3	$g_{1,7}$
$g_{1,8}$	$L_x + K_y, K_z$	2	2	$g_{1,7}$
$g_{2,0}$	L_x, P_x, P_y, P_z, P_t	5	4	$g_{2,0}$
$g_{2,1}$	$L_x, P_x + \alpha P_t, P_y, P_z$	4	3	$g_{2,0}$
$g_{2,2}$	L_x, P_x, P_y, P_z	4	3	$g_{2,0}$
$g_{2,3}$	L_x, P_y, P_z, P_t	4	3	$g_{2,0}$
$g_{2,4}$	L_x, P_x, P_t	3	3	$g_{2,4}$
$g_{2,5}$	L_x, P_y, P_z	3	2	$g_{2,0}$
$g_{2,6}$	$L_x, P_x + \alpha P_t$	2	2	$g_{2,4}$

$g_{2,7}$	L_x, P_z	2	2	$g_{2,4}$
$g_{2,8}$	L_x, P_t	2	2	$g_{2,4}$
$g_{2,9}$	L_x	1	1	$g_{2,4}$
$g_{3,0}$	K_y, P_x, P_y, P_z, P_t	5	4	$g_{3,0}$
$g_{3,1}$	$K_y, P_x, P_y + \alpha P_t, P_z$	4	3	$g_{3,0}$
$g_{3,2}$	K_y, P_x, P_y, P_z	4	3	$g_{3,0}$
$g_{3,3}$	K_y, P_x, P_z, P_t	4	3	$g_{3,0}$
$g_{3,4}$	$K_y, P_x + P_z, P_y, P_t$	4	4	$g_{3,4}$
$g_{3,5}$	$K_y, P_x + P_z, P_y + \alpha P_t$	3	3	$g_{3,4}$
$g_{3,6}$	$K_y, P_x + P_z, P_y$	3	3	$g_{3,4}$
$g_{3,7}$	K_y, P_x, P_z	3	2	$g_{3,0}$
$g_{3,8}$	$K_y, P_x + P_z, P_t$	3	3	$g_{3,4}$
$g_{3,9}$	K_y, P_y, P_t	3	3	$g_{3,9}$
$g_{3,10}$	$K_y, P_x + P_z$	2	2	$g_{3,4}$
$g_{3,11}$	$K_y, P_y + \alpha P_t$	2	2	$g_{3,9}$
$g_{3,12}$	K_y, P_y	2	2	$g_{3,9}$
$g_{3,13}$	K_y, P_t	2	2	$g_{3,9}$
$g_{3,14}$	K_y	1	1	$g_{3,9}$
$g_{4,0}$	$L_x + K_y, P_x, P_y, P_z, P_t$	5	4	$g_{4,0}$
$g_{4,1}$	$L_x + K_y, P_x + P_y, P_y + \alpha P_t, P_z$	4	3	$g_{4,0}$
$g_{4,2}$	$L_x + K_y, P_x, P_y, P_z$	4	3	$g_{4,0}$
$g_{4,3}$	$L_x + K_y, P_x + P_y, P_z, P_t$	4	3	$g_{4,0}$
$g_{4,4}$	$L_x + K_y, P_x + P_y, P_z + \alpha P_t$	3	3	$g_{4,3}$
$g_{4,5}$	$L_x + K_y, P_x + P_y, P_z$	3	2	$g_{4,0}$
$g_{4,6}$	$L_x + K_y, P_x + P_y, P_t$	3	3	$g_{4,3}$
$g_{4,7}$	$L_x + K_y, P_x + P_y + \alpha P_t$	2	2	$g_{4,6}$

$g_{4,8}$	$L_x + K_y, P_x + P_y$	2	2	$g_{4,3}$	
$g_{4,9}$	$L_x + K_y, P_t$	2	2	$g_{4,6}$	
$g_{4,10}$	$L_x + K_y$	1	1	$g_{4,6}$	
$g_{5,0}$	P_x, P_y, P_z, P_t	4	4	$g_{0,0}$	
$g_{5,1}$	P_x, P_y, P_z	3	3	$g_{0,0}$	
$g_{5,2}$	$P_x, P_z, P_t + \alpha P_y$	3	3	$g_{3,0}$	
$g_{5,3}$	P_x, P_z, P_t	3	3	$g_{3,0}$	
$g_{5,4}$	$P_y, P_z, P_t + \alpha P_x$	3	3	$g_{2,0}$	
$g_{5,5}$	P_y, P_z, P_t	3	3	$g_{2,0}$	
$g_{5,6}$	$P_x + P_y, P_z, P_t + \alpha P_y$	3	3	$g_{4,0}$	
$g_{5,7}$	$P_x + P_y, P_z, P_t$	3	3	$g_{4,0}$	
$g_{5,8}$	$P_z, P_t + \alpha P_y$	2	2	$g_{5,0}$	
$g_{5,9}$	P_z, P_t	2	2	$g_{2,0}$	
$g_{5,10}$	$P_y, P_t + \alpha P_x$	2	2	$g_{5,0}$	
$g_{5,11}$	P_y, P_t	2	2	$g_{3,0}$	
$g_{5,12}$	$P_z, P_t + \alpha(P_x + P_y)$	2	2	$g_{5,0}$	
$g_{5,13}$	$P_x + P_z, P_t + \alpha P_y + \beta P_z$	2	2	$g_{5,0}$	
$g_{5,14}$	$P_x + P_z, P_t + \alpha P_y$	2	2	$g_{5,0}$	
$g_{5,15}$	$P_x + P_z, P_t + \alpha P_z$	2	2	$g_{5,0}$	
$g_{5,16}$	$P_x + P_y, P_t$	2	2	$g_{1,0}$	
$g_{5,17}$	$P_y, P_t + \alpha P_z$	2	2	$g_{5,0}$	
$g_{5,18}$	$P_x + \alpha P_y, P_z$	2	2	$g_{1,0}$	*for* $\alpha = 1$
				$g_{5,0}$	*for* $\alpha \neq 1$
$g_{5,19}$	P_x, P_z	2	2	$g_{3,0}$	
$g_{5,20}$	P_y, P_z	2	2	$g_{2,0}$	
$g_{5,21}$	$P_x + \alpha P_t$	1	1	$g_{2,0}$	
$g_{5,22}$	P_x	1	1	$g_{2,0}$	

$g_{5,23}$	$P_y + \alpha P_t$	1	1	$g_{3,0}$
$g_{5,24}$	P_y	1	1	$g_{3,0}$
$g_{5,25}$	$P_x + P_y + \alpha P_t$	1	1	$g_{4,0}$
$g_{5,26}$	$P_x + P_y$	1	1	$g_{1,0}$
$g_{5,27}$	P_t	1	1	$g_{0,0}$
$g_{5,28}$	0	0	0	$g_{0,0}$
$\tilde{g}_{1,9}$	$L_x + K_y, K_z + \alpha P_t, P_x, P_y, P_z$	5	4	$g_{1,0}$
$\tilde{g}_{1,10}$	$L_x + K_y, K_z + \alpha P_t, P_x + P_y, P_z$	4	3	$g_{1,0}$
$\tilde{g}_{1,11}$	$L_x + K_y, K_z + \alpha P_z, P_x + P_y, P_t$	4	4	$g_{1,2}$
$\tilde{g}_{1,12}$	$L_x + K_y, K_z + \alpha P_z, P_x + P_y, P_t + \beta P_z$	4	4	$g_{1,2}$
$\tilde{g}_{1,13}$	$L_x + K_y, K_z + \alpha P_z + \beta P_t, P_x + P_y$	3	3	$g_{1,2}$
$\tilde{g}_{1,14}$	$L_x + K_y, K_z + \alpha P_z, P_x + P_y$	3	3	$g_{1,2}$
$\tilde{g}_{1,15}$	$L_x + K_y, K_z + \alpha P_t, P_x + P_y$	3	3	$g_{1,2}$
$\tilde{g}_{1,16}$	$L_x + K_y, K_z + \alpha P_t$	2	2	$g_{1,7}$
$\tilde{g}_{2,10}$	$L_x + \alpha P_t, P_x + \beta P_t, P_y, P_z$	4	4	$g_{2,0}$
$\tilde{g}_{2,11}$	$L_x + \alpha P_t, P_x, P_y, P_z$	4	4	$g_{2,0}$
$\tilde{g}_{2,12}$	$L_x + \alpha P_x, P_y, P_z, P_t$	4	4	$g_{2,0}$
$\tilde{g}_{2,13}$	$L_x + \alpha P_x + \beta P_t, P_y, P_z$	3	3	$g_{2,0}$
$\tilde{g}_{2,14}$	$L_x + \alpha P_x, P_y, P_z$	3	3	$g_{2,0}$
$\tilde{g}_{2,15}$	$L_x + \alpha P_t, P_y, P_z$	3	3	$g_{2,0}$
$\tilde{g}_{2,16}$	$L_x + \alpha P_t, P_x + \beta P_t$	2	2	$g_{2,4}$
$\tilde{g}_{2,17}$	$L_x + \alpha P_t, P_x$	2	2	$g_{2,4}$
$\tilde{g}_{2,18}$	$L_x + \alpha P_x, P_t$	2	2	$g_{2,4}$
$\tilde{g}_{2,19}$	$L_x + \alpha P_x + \beta P_t$	1	1	$g_{2,4}$
$\tilde{g}_{2,20}$	$L_x + \alpha P_x$	1	1	$g_{2,4}$
$\tilde{g}_{2,21}$	$L_x + \alpha P_t$	1	1	$g_{2,4}$

$\tilde{g}_{3,15}$	$K_y + \alpha P_t, P_x, P_y + \beta P_t, P_z$	4	4	$g_{3,0}$
$\tilde{g}_{3,16}$	$K_y + \alpha P_t, P_x, P_y, P_z$	4	4	$g_{3,0}$
$\tilde{g}_{3,17}$	$K_y + \alpha P_y, P_x, P_z, P_t$	4	4	$g_{3,0}$
$\tilde{g}_{3,18}$	$K_y + \alpha P_t, P_x + P_z, P_y + \beta P_t$	3	3	$g_{3,4}$
$\tilde{g}_{3,19}$	$K_y + \alpha P_t, P_x + P_z, P_y$	3	3	$g_{3,4}$
$\tilde{g}_{3,20}$	$K_y + \alpha P_y + \beta P_t, P_x, P_z$	3	3	$g_{3,0}$
$\tilde{g}_{3,21}$	$K_y + \alpha P_y, P_x, P_z$	3	3	$g_{3,0}$
$\tilde{g}_{3,22}$	$K_y + \alpha P_t, P_x, P_z$	3	3	$g_{3,0}$
$\tilde{g}_{3,23}$	$K_y + \alpha P_y, P_x + P_z, P_t$	3	3	$g_{3,4}$
$\tilde{g}_{3,24}$	$K_y + \alpha P_y + \beta P_t, P_x + P_z$	2	2	$g_{3,4}$
$\tilde{g}_{3,25}$	$K_y + \alpha P_y, P_x + P_z$	2	2	$g_{3,4}$
$\tilde{g}_{3,26}$	$K_y + \alpha P_t, P_x + P_z$	2	2	$g_{3,4}$
$\tilde{g}_{3,27}$	$K_y + \alpha P_t, P_y + \beta P_t$	2	2	$g_{3,9}$
$\tilde{g}_{3,28}$	$K_y + \alpha P_t, P_y$	2	2	$g_{3,9}$
$\tilde{g}_{3,29}$	$K_y + \alpha P_y, P_t$	2	2	$g_{3,9}$
$\tilde{g}_{3,30}$	$K_y + \alpha P_y + \beta P_t$	1	1	$g_{3,9}$
$\tilde{g}_{3,31}$	$K_y + \alpha P_y$	1	1	$g_{3,9}$
$\tilde{g}_{3,32}$	$K_y + \alpha P_t$	1	1	$g_{3,9}$

$\tilde{g}_{4,11}$	$L_x + K_y + \alpha P_t, P_x + P_y, P_y + \beta P_t, P_z$	4	4	$g_{4,0}$
$\tilde{g}_{4,12}$	$L_x + K_y + \alpha P_t, P_x, P_y, P_z$	4	4	$g_{4,0}$
$\tilde{g}_{4,13}$	$L_x + K_y + \alpha P_x, P_x + P_y, P_z + \beta P_t$	3	3	$g_{5,7}$
$\tilde{g}_{4,14}$	$L_x + K_y + \alpha P_x + \beta P_t, P_x + P_y, P_z$	3	3	$g_{4,0}$
$\tilde{g}_{4,15}$	$L_x + K_y + \alpha P_x, P_x + P_y, P_z$	3	3	$g_{4,0}$
$\tilde{g}_{4,16}$	$L_x + K_y + \alpha P_t, P_x + P_y, P_z$	3	3	$g_{4,0}$
$\tilde{g}_{4,17}$	$L_x + K_y + \alpha P_x, P_x + P_y, P_t$	3	3	$g_{5,7}$
$\tilde{g}_{4,18}$	$L_x + K_y + \alpha P_x + \beta P_t, P_x + P_y$	2	2	$g_{5,7}$
$\tilde{g}_{4,19}$	$L_x + K_y + \alpha P_x, P_x + P_y$	2	2	$g_{5,7}$

$\tilde{g}_{4,20}$	$L_x + K_y + \alpha P_t, P_x + P_y$	2	2	$g_{4,3}$
$\tilde{g}_{4,21}$	$L_x + K_y + \alpha P_x, P_t$	2	2	$g_{5,16}$
$\tilde{g}_{4,22}$	$L_x + K_y + \alpha P_x + \beta P_t$	1	1	$g_{5,16}$
$\tilde{g}_{4,23}$	$L_x + K_y + \alpha P_x$	1	1	$g_{5,16}$
$\tilde{g}_{4,24}$	$L_x + K_y + \alpha P_t$	1	1	$g_{5,16}$

Algebras containing the scaling operators D_1 and D_2 are algebraically isomorphic and differ just by the action of the scaling operator in the same way as it happened in the case of Eq. (3.1). Therefore, the algebraic structure of only one of them is presented in Table 3.7.

Table 3.7: The representatives of conjugacy classes of the algebra generated by $(P_x, P_y, P_z, P_t, L_x, K_y, K_z, D)$; α and β are different from 0; $\epsilon = \pm 1$; n is the dimension of an algebra; n_1 is the dimension of the corresponding orbit.

Number	Generators	n	n_1	Normalizer
$g_{0,0}$	$L_x, K_y, K_z, P_x, P_y, P_z, P_t, D$	8	5	$g_{0,0}$
$g_{0,1}$	$L_x, K_y, K_z, P_x, P_y, P_z, P_t$	7	4	$g_{0,0}$
$g_{0,2}$	$L_x, K_y, K_z, P_x, P_y, P_z, D$	7	4	$g_{0,2}$
$g_{0,3}$	$L_x, K_y, K_z, P_x, P_y, P_z$	6	3	$g_{0,0}$
$g_{0,4}$	L_x, K_y, K_z, P_t, D	5	4	$g_{0,4}$
$g_{0,5}$	L_x, K_y, K_z, P_t	4	3	$g_{0,4}$
$g_{0,6}$	L_x, K_y, K_z, D	4	3	$g_{0,6}$
$g_{0,7}$	L_x, K_y, K_z	3	2	$g_{0,4}$
$g_{1,0}$	$L_x + K_y, K_z, P_x, P_y, P_z, P_t, D$	7	5	$g_{1,0}$
$g_{1,1}$	$L_x + K_y, K_z, P_x, P_y, P_z, P_t$	6	4	$g_{1,0}$
$g_{1,2}$	$L_x + K_y, K_z, P_x, P_y, P_z, D$	6	4	$g_{1,2}$

$g_{1,3}$	$L_x + K_y, K_z, P_x + P_y, P_z, P_t, D$	6	5	$g_{1,3}$
$g_{1,4}$	$L_x + K_y, K_z, P_x, P_y, P_z$	5	3	$g_{1,0}$
$g_{1,5}$	$L_x + K_y, K_z, P_x + P_y, P_z, P_t$	5	4	$g_{1,3}$
$g_{1,6}$	$L_x + K_y, K_z, P_x + P_y, P_z, D$	5	4	$g_{1,6}$
$g_{1,7}$	$L_x + K_y, K_z, P_x + P_y, P_t, D$	5	5	$g_{1,7}$
$g_{1,8}$	$L_x + K_y, K_z, P_x + P_y, P_t + \alpha P_z$	4	4	$g_{1,5}$
$g_{1,9}$	$L_x + K_y, K_z, P_x + P_y, P_t$	4	4	$g_{1,3}$
$g_{1,10}$	$L_x + K_y, K_z, P_x + P_y, P_z$	4	3	$g_{1,3}$
$g_{1,11}$	$L_x + K_y, K_z, P_x + P_y, D$	4	4	$g_{1,11}$
$g_{1,12}$	$L_x + K_y, K_z, P_t, D$	4	4	$g_{1,12}$

$g_{1,13}$	$L_x + K_y, K_z, P_x + P_y$	3	3	$g_{1,3}$
$g_{1,14}$	$L_x + K_y, K_z, P_t$	3	3	$g_{1,12}$
$g_{1,15}$	$L_x + K_y, K_z, D$	3	3	$g_{1,15}$
$g_{1,16}$	$L_x + K_y, K_z$	2	2	$g_{1,12}$

$g_{2,0}$	$L_x, P_x, P_y, P_z, P_t, D$	6	5	$g_{2,0}$
$g_{2,1}$	L_x, P_x, P_y, P_z, P_t	5	4	$g_{2,0}$
$g_{2,2}$	L_x, P_x, P_y, P_z, D	5	4	$g_{2,2}$
$g_{2,3}$	L_x, P_y, P_z, P_t, D	5	4	$g_{2,3}$
$g_{2,4}$	$L_x, P_x + \alpha P_t, P_y, P_z$	4	3	$g_{2,1}$
$g_{2,5}$	L_x, P_x, P_y, P_z	4	3	$g_{2,0}$
$g_{2,6}$	L_x, P_y, P_z, P_t	4	3	$g_{2,0}$
$g_{2,7}$	L_x, P_x, P_t, D	4	4	$g_{2,7}$
$g_{2,8}$	L_x, P_y, P_z, D	4	3	$g_{2,8}$
$g_{2,9}$	L_x, P_x, P_t	3	3	$g_{2,7}$
$g_{2,10}$	L_x, P_y, P_z	3	2	$g_{2,0}$
$g_{2,11}$	L_x, P_x, D	3	3	$g_{2,11}$

$g_{2,12}$	L_x, P_t, D	3	3	$g_{2,12}$
$g_{2,13}$	$L_x, P_x + \alpha P_t$	2	2	$g_{2,9}$
$g_{2,14}$	L_x, P_x	2	2	$g_{2,7}$
$g_{2,15}$	L_x, P_t	2	2	$g_{2,7}$
$g_{2,16}$	L_x, D	2	2	$g_{2,16}$
$g_{2,17}$	L_x	1	1	$g_{2,7}$
$g_{3,0}$	$K_y, P_x, P_y, P_z, P_t, D$	6	5	$g_{3,0}$
$g_{3,1}$	K_y, P_x, P_y, P_z, P_t	5	4	$g_{3,0}$
$g_{3,2}$	K_y, P_x, P_y, P_z, D	5	4	$g_{3,2}$
$g_{3,3}$	K_y, P_x, P_z, P_t, D	5	4	$g_{3,3}$
$g_{3,4}$	$K_y, P_x + P_z, P_y, P_t, D$	5	5	$g_{3,4}$
$g_{3,5}$	$K_y, P_x, P_y + \alpha P_t, P_z$	4	3	$g_{3,1}$
$g_{3,6}$	K_y, P_x, P_y, P_z	4	3	$g_{3,0}$
$g_{3,7}$	K_y, P_x, P_z, P_t	4	3	$g_{3,0}$
$g_{3,8}$	$K_y, P_x + P_z, P_y, P_t$	4	4	$g_{3,4}$
$g_{3,9}$	K_y, P_x, P_z, D	4	3	$g_{3,9}$
$g_{3,10}$	$K_y, P_x + P_z, P_y, D$	4	4	$g_{3,10}$
$g_{3,11}$	$K_y, P_x + P_z, P_t, D$	4	4	$g_{3,11}$
$g_{3,12}$	K_y, P_y, P_t, D	4	4	$g_{3,12}$
$g_{3,13}$	$K_y, P_x + P_z, P_y + \alpha P_t$	3	3	$g_{3,8}$
$g_{3,14}$	$K_y, P_x + P_z, P_y$	3	3	$g_{3,4}$
$g_{3,15}$	K_y, P_x, P_z	3	2	$g_{3,0}$
$g_{3,16}$	$K_y, P_x + P_z, P_t$	3	3	$g_{3,4}$
$g_{3,17}$	K_y, P_y, P_t	3	3	$g_{3,12}$
$g_{3,18}$	$K_y, P_x + P_z, D$	3	3	$g_{3,18}$
$g_{3,19}$	K_y, P_y, D	3	3	$g_{3,19}$
$g_{3,20}$	K_y, P_t, D	3	3	$g_{3,20}$

$g_{3,21}$	$K_y, P_x + P_z$	2	2	$g_{3,4}$
$g_{3,22}$	$K_y, P_y + \alpha P_t$	2	2	$g_{3,17}$
$g_{3,23}$	K_y, P_y	2	2	$g_{3,12}$
$g_{3,24}$	K_y, P_t	2	2	$g_{3,12}$
$g_{3,25}$	K_y, D	2	2	$g_{3,25}$
$g_{3,26}$	K_y	1	1	$g_{3,12}$
$g_{4,0}$	$L_x + K_y, P_x, P_y, P_z, P_t, D$	6	5	$g_{4,0}$
$g_{4,1}$	$L_x + K_y, P_x, P_y, P_z, P_t$	5	4	$g_{4,0}$
$g_{4,2}$	$L_x + K_y, P_x, P_y, P_z, D$	5	4	$g_{4,2}$
$g_{4,3}$	$L_x + K_y, P_x + P_y, P_z, P_t, D$	5	4	$g_{4,3}$
$g_{4,4}$	$L_x + K_y, P_x + P_y, P_t + \alpha P_y, P_z$	4	3	$\tilde{g}_{1,17}\|_{\alpha=-1}$
$g_{4,5}$	$L_x + K_y, P_x, P_y, P_z$	4	3	$g_{4,0}$
$g_{4,6}$	$L_x + K_y, P_x + P_y, P_z, P_t$	4	3	$g_{4,3}$
$g_{4,7}$	$L_x + K_y, P_x + P_y, P_z, D$	4	3	$g_{4,7}$
$g_{4,8}$	$L_x + K_y, P_x + P_y, P_t, D$	4	4	$g_{4,8}$
$g_{4,9}$	$L_x + K_y, P_x + P_y, P_z + \alpha P_t$	3	3	$g_{4,6}$
$g_{4,10}$	$L_x + K_y, P_x + P_y, P_z$	3	2	$g_{4,0}$
$g_{4,11}$	$L_x + K_y, P_x + P_y, P_t$	3	3	$g_{4,3}$
$g_{4,12}$	$L_x + K_y, P_x + P_y, D$	3	3	$g_{4,12}$
$g_{4,13}$	$L_x + K_y, P_t, D$	3	3	$g_{4,13}$
$g_{4,14}$	$L_x + K_y, P_x + P_y + \alpha P_t$	2	2	$g_{4,11}$
$g_{4,15}$	$L_x + K_y, P_x + P_y$	2	2	$g_{4,3}$
$g_{4,15}$	$L_x + K_y, P_t$	2	2	$g_{4,8}$
$g_{4,16}$	$L_x + K_y, D$	2	2	$g_{4,16}$
$g_{4,17}$	$L_x + K_y$	1	1	$g_{4,8}$
$g_{5,0}$	P_x, P_y, P_z, P_t, D	5	5	$g_{0,0}$

$g_{5,1}$	P_x, P_y, P_z, P_t	4	4	$g_{0,0}$
$g_{5,2}$	P_x, P_y, P_z, D	4	4	$g_{0,2}$
$g_{5,3}$	P_x, P_z, P_t, D	4	4	$g_{3,3}$
$g_{5,4}$	P_y, P_z, P_t, D	4	4	$g_{2,3}$
$g_{5,5}$	$P_x + P_y, P_z, P_t, D$	4	4	$g_{1,3}$
$g_{5,6}$	P_x, P_y, P_z	3	3	$g_{0,0}$
$g_{5,7}$	$P_x, P_z, P_t + \alpha P_y$	3	3	$g_{3,1}$
$g_{5,8}$	P_x, P_z, P_t	3	3	$g_{3,0}$
$g_{5,9}$	$P_y, P_z, P_t + \alpha P_x$	3	3	$g_{2,1}$
$g_{5,10}$	P_y, P_z, P_t	3	3	$g_{2,0}$
$g_{5,11}$	$P_x + P_z, P_y, P_t + \alpha P_z$	3	3	$g_{3,1}$
$g_{5,12}$	$P_x + P_y, P_z, P_t$	3	3	$g_{1,0}$
$g_{5,13}$	P_x, P_z, D	3	3	$g_{3,9}$
$g_{5,14}$	P_y, P_z, D	3	3	$g_{2,8}$
$g_{5,15}$	$P_x + P_y, P_z, D$	3	3	$g_{1,6}$
$g_{5,16}$	P_x, P_t, D	3	3	$g_{2,7}$
$g_{5,17}$	P_y, P_t, D	3	3	$g_{3,12}$
$g_{5,18}$	$P_x + P_y, P_t, D$	3	3	$g_{1,7}$
$g_{5,19}$	$P_x, P_t + \alpha P_y$	2	2	$g_{5,1}$
$g_{5,20}$	P_x, P_t	2	2	$g_{2,0}$
$g_{5,21}$	$P_y, P_t + \alpha P_x$	2	2	$g_{5,1}$
$g_{5,22}$	P_y, P_t	2	2	$g_{3,0}$
$g_{5,23}$	$P_y, P_t + \alpha(P_x + P_z)$	2	2	$g_{5,1}$
$g_{5,24}$	$P_x + P_z, P_t + \alpha P_y + \beta P_z$	2	2	$g_{5,1}$
$g_{5,25}$	$P_x + P_y, P_t + \alpha P_z$	2	2	$g_{1,5}$
$g_{5,26}$	$P_x + P_z, P_t + \alpha P_z$	2	2	$g_{5,1}$
$g_{5,27}$	$P_x + P_y, P_t$	2	2	$g_{1,0}$
$g_{5,28}$	$P_y, P_t + \alpha P_z$	2	2	$g_{5,1}$

$g_{5,29}$	$P_x + \alpha P_y, P_z$	2	2	$g_{1,0}$ $for\ \alpha = 1$ $g_{5,0}$ $for\ \alpha \neq 1$
$g_{5,30}$	P_x, P_z	2	2	$g_{3,0}$
$g_{5,31}$	P_y, P_z	2	2	$g_{2,0}$
$g_{5,32}$	P_x, D	2	2	$g_{2,11}$
$g_{5,33}$	P_y, D	2	2	$g_{3,19}$
$g_{5,34}$	P_t, D	2	2	$g_{0,4}$
$g_{5,35}$	$P_x + P_y, D$	2	2	$g_{1,11}$
$g_{5,36}$	$P_x + \alpha P_t$	1	1	$g_{5,1}$
$g_{5,37}$	P_x	1	1	$g_{2,0}$
$g_{5,38}$	$P_y + \alpha P_t$	1	1	$g_{3,1}$
$g_{5,39}$	P_y	1	1	$g_{3,0}$
$g_{5,40}$	$P_x + P_z + \alpha P_t$	1	1	$g_{5,1}$
$g_{5,41}$	$P_x + P_y$	1	1	$g_{1,0}$
$g_{5,42}$	P_t	1	1	$g_{0,0}$
$g_{5,43}$	D	1	1	$g_{0,6}$
$g_{5,44}$	0	0	0	$g_{0,0}$
$\tilde{g}_{1,17}$	$L_x + K_y, K_z + \alpha D, P_x, P_y, P_z, P_t$	6	5	$g_{1,0}$
$\tilde{g}_{1,18}$	$L_x + K_y, K_z + \alpha P_t, P_x, P_y, P_z$	5	4	$g_{1,1}$
$\tilde{g}_{1,19}$	$L_x + K_y + \alpha P_t, K_z + D/2, P_x, P_y, P_z$	5	5	$g_{1,4}$
$\tilde{g}_{1,20}$	$L_x + K_y, K_z + \alpha D, P_x, P_y, P_z$	5	4	$g_{1,2}$
$\tilde{g}_{1,21}$	$L_x + K_y, K_z - D, P_x + P_y, P_z, P_t + \alpha P_y$	5	4	$\tilde{g}_{1,21}$
$\tilde{g}_{1,22}$	$L_x + K_y, K_z + \alpha D, P_x + P_y, P_z, P_t$	5	4	$g_{1,3}$
$\tilde{g}_{1,23}$	$L_x + K_y + \alpha P_x, K_z + 2D, P_x + P_y, P_z, P_t$	5	5	$g_{1,23}$
$\tilde{g}_{1,24}$	$L_x + K_y, K_z + \alpha P_t, P_x + P_y, P_z$	4	3	$g_{1,5}$
$\tilde{g}_{1,25}$	$L_x + K_y, K_z + \alpha P_z, P_x + P_y, P_t$	4	4	$g_{1,5}$
$\tilde{g}_{1,26}$	$L_x + K_y, K_z + \alpha P_z, P_x + P_y, P_t + \beta P_z$	4	4	$g_{1,5}$

$\tilde{g}_{1,27}$	$L_x + K_y + \alpha P_t, K_z + D/2, P_x + P_y, P_z$	4	4	$g_{1,27}$
$\tilde{g}_{1,28}$	$L_x + K_y + \alpha P_x, K_z + 2D, P_x + P_y, P_z$	4	4	$g_{1,28}$
$\tilde{g}_{1,29}$	$L_x + K_y, K_z + \alpha D, P_x + P_y, P_z$	4	3	$g_{1,6}$
$\tilde{g}_{1,30}$	$L_x + K_y, K_z + \alpha P_x + D, P_x + P_y, P_z$	4	3	$g_{1,10}$
$\tilde{g}_{1,31}$	$L_x + K_y + \alpha P_x, K_z + 2D, P_x + P_y, P_t$	4	4	$\tilde{g}_{1,31}$
$\tilde{g}_{1,32}$	$L_x + K_y, K_z + \alpha D, P_x + P_y, P_t$	4	4	$g_{1,7}$
$\tilde{g}_{1,33}$	$L_x + K_y + \alpha P_z, K_z + D, P_x + P_y, P_t$	4	4	$\tilde{g}_{1,33}$
$\tilde{g}_{1,34}$	$L_x + K_y, K_z + \epsilon D + P_y, P_x + P_y, P_z$	4	3	$\tilde{g}_{1,20}\rvert_{\alpha=\epsilon}$
$\tilde{g}_{1,35}$	$L_x + K_y + \epsilon P_z/2, K_z - D + \epsilon P_y, P_x + P_y, P_t$	4	4	$g_{1,35}$
$\tilde{g}_{1,36}$	$L_x + K_y, K_z + \alpha P_z + \beta P_t, P_x + P_y$	3	3	$g_{1,5}$
$\tilde{g}_{1,37}$	$L_x + K_y, K_z + \alpha P_z, P_x + P_y$	3	3	$g_{1,5}$
$\tilde{g}_{1,38}$	$L_x + K_y, K_z + \alpha P_t, P_x + P_y$	3	3	$g_{1,5}$
$\tilde{g}_{1,39}$	$L_x + K_y, K_z + D, P_x + P_y + \alpha P_t$	3	3	$\tilde{g}_{1,39}$
$\tilde{g}_{1,40}$	$L_x + K_y, K_z + \alpha D, P_x + P_y$	3	3	$g_{1,11}$
$\tilde{g}_{1,41}$	$L_x + K_y + \alpha P_t, K_z + D/2, P_x + P_y$	3	3	$\tilde{g}_{1,41}$
$\tilde{g}_{1,42}$	$L_x + K_y + \epsilon(P_x - P_y), K_z + 2D, P_t$	3	3	$\tilde{g}_{1,42}$
$\tilde{g}_{1,43}$	$L_x + K_y, K_z + \alpha D, P_t$	3	3	$g_{1,12}$
$\tilde{g}_{1,44}$	$L_x + K_y + \alpha P_x, K_z + 2D, P_x + P_y$	3	3	$\tilde{g}_{1,44}$
$\tilde{g}_{1,45}$	$L_x + K_y + \alpha P_z, K_z + D, P_x + P_y$	3	3	$\tilde{g}_{1,45}$
$\tilde{g}_{1,46}$	$L_x + K_y + \alpha P_z, K_z + D, P_t$	3	3	$\tilde{g}_{1,46}$
$\tilde{g}_{1,47}$	$L_x + K_y + \epsilon P_z/2, K_z - D + \epsilon P_y, P_x + P_y$	3	3	$\tilde{g}_{1,47}$
$\tilde{g}_{1,48}$	$L_x + K_y + \epsilon P_z/2, K_z - D + \epsilon P_y, P_t$	3	3	$\tilde{g}_{1,48}$
$\tilde{g}_{1,49}$	$L_x + K_y, K_z + \alpha P_t$	2	2	$g_{1,9}$
$\tilde{g}_{1,50}$	$L_x + K_y + \alpha P_t, K_z + D/2$	2	2	$\tilde{g}_{1,50}$
$\tilde{g}_{1,51}$	$L_x + K_y + \alpha P_z, K_z + D$	2	2	$\tilde{g}_{1,51}$
$\tilde{g}_{1,52}$	$L_x + K_y + \alpha(P_x - P_y), K_z + 2D$	2	2	$\tilde{g}_{1,52}$
$\tilde{g}_{1,53}$	$L_x + K_y, K_z + \alpha D$	2	2	$\tilde{g}_{1,53}$
$\tilde{g}_{1,54}$	$L_x + K_y + \epsilon P_z/2, K_z - D + \epsilon P_y$	2	2	$\tilde{g}_{1,54}$

$\tilde{g}_{2,17}$	$L_x + \alpha D, P_x, P_z, P_y, P_t$	5	5	$g_{2,0}$
$\tilde{g}_{2,18}$	$L_x + \alpha P_t, P_x + \beta P_t, P_y, P_z$	4	4	$g_{2,1}$
$\tilde{g}_{2,19}$	$L_x + \alpha P_t, P_x, P_y, P_z$	4	4	$g_{2,1}$
$\tilde{g}_{2,20}$	$L_x + \alpha P_x, P_y, P_z, P_t$	4	4	$g_{2,1}$
$\tilde{g}_{2,21}$	$L_x + \alpha D, P_x, P_y, P_z$	4	4	$g_{2,2}$
$\tilde{g}_{2,22}$	$L_x + \alpha D, P_y, P_z, P_t$	4	4	$g_{2,3}$
$\tilde{g}_{2,23}$	$L_x + \alpha P_x + \beta P_t, P_y, P_z$	3	3	$g_{2,1}$
$\tilde{g}_{2,24}$	$L_x + \alpha P_x, P_y, P_z$	3	3	$g_{2,1}$
$\tilde{g}_{2,25}$	$L_x + \alpha P_t, P_y, P_z$	3	3	$g_{2,1}$
$\tilde{g}_{2,26}$	$L_x + \alpha D, P_x, P_t$	3	3	$g_{2,7}$
$\tilde{g}_{2,27}$	$L_x + \alpha D, P_y, P_z$	3	3	$g_{2,8}$
$\tilde{g}_{2,28}$	$L_x + \alpha P_t, P_x + \beta P_t$	2	2	$g_{2,9}$
$\tilde{g}_{2,29}$	$L_x + \alpha P_t, P_x$	2	2	$g_{2,9}$
$\tilde{g}_{2,30}$	$L_x + \alpha P_x, P_t$	2	2	$g_{2,9}$
$\tilde{g}_{2,31}$	$L_x + \alpha D, P_x$	2	2	$g_{2,11}$
$\tilde{g}_{2,32}$	$L_x + \alpha D, P_t$	2	2	$g_{2,12}$
$\tilde{g}_{2,33}$	$L_x + \alpha P_x + \beta P_t$	1	1	$g_{2,9}$
$\tilde{g}_{2,34}$	$L_x + \alpha P_x$	1	1	$g_{2,9}$
$\tilde{g}_{2,35}$	$L_x + \alpha P_t$	1	1	$g_{2,9}$
$\tilde{g}_{2,36}$	$L_x + \alpha D$	1	1	$g_{2,16}$

$\tilde{g}_{3,27}$	$K_y + \alpha D, P_x, P_y, P_z, P_t$	5	5	$g_{3,0}$
$\tilde{g}_{3,28}$	$K_y + \alpha P_t, P_x, P_y + \beta P_t, P_z$	4	4	$g_{3,1}$
$\tilde{g}_{3,29}$	$K_y + \alpha P_t, P_x, P_y, P_z$	4	4	$g_{3,1}$
$\tilde{g}_{3,30}$	$K_y + \alpha P_y, P_x, P_z, P_t$	4	4	$g_{3,1}$
$\tilde{g}_{3,31}$	$K_y + \alpha D, P_x, P_y, P_z$	4	4	$g_{3,2}$
$\tilde{g}_{3,32}$	$K_y + \alpha D, P_x + P_z, P_y, P_t$	4	4	$g_{3,4}$
$\tilde{g}_{3,33}$	$K_y + \alpha D, P_x, P_z, P_t$	4	4	$g_{3,3}$

$\tilde{g}_{3,34}$	$K_y + D, P_x - P_z, P_y, P_z + \alpha P_t$	4	4	$\tilde{g}_{3,34}$
$\tilde{g}_{3,35}$	$K_y + D + \epsilon P_z, P_x + P_z, P_y, P_t$	4	4	$\tilde{g}_{3,35}$
$\tilde{g}_{3,36}$	$K_y + \alpha P_t, P_x + P_z, P_y + \beta P_t$	3	3	$g_{3,8}$
$\tilde{g}_{3,37}$	$K_y + \alpha P_t, P_x + P_z, P_y$	3	3	$g_{3,8}$
$\tilde{g}_{3,38}$	$K_y + \alpha P_y + \beta P_t, P_x, P_z$	3	3	$g_{3,1}$
$\tilde{g}_{3,39}$	$K_y + \alpha P_y, P_x, P_z$	3	3	$g_{3,1}$
$\tilde{g}_{3,40}$	$K_y + \alpha P_t, P_x, P_z$	3	3	$g_{3,1}$
$\tilde{g}_{3,41}$	$K_y + \alpha P_y, P_x + P_z, P_t$	3	3	$g_{3,8}$
$\tilde{g}_{3,42}$	$K_y + \alpha D, P_x + P_z, P_y$	3	3	$g_{3,10}$
$\tilde{g}_{3,43}$	$K_y + \alpha D, P_x, P_z$	3	3	$g_{3,9}$
$\tilde{g}_{3,44}$	$K_y + \alpha D, P_x + P_z, P_t$	3	3	$g_{3,11}$
$\tilde{g}_{3,45}$	$K_y + \alpha D, P_y, P_t$	3	3	$g_{3,12}$
$\tilde{g}_{3,46}$	$K_y + D, P_x + P_z + \alpha P_t, P_y$	3	3	$\tilde{g}_{3,46}$
$\tilde{g}_{3,47}$	$K_y + D, P_x - P_z, P_z + \alpha P_t$	3	3	$\tilde{g}_{3,47}$
$\tilde{g}_{3,48}$	$K_y + D + \epsilon P_z, P_x + P_z, P_t$	3	3	$\tilde{g}_{3,48}$
$\tilde{g}_{3,49}$	$K_y + D + \epsilon P_z, P_x + P_z, P_y$	3	3	$\tilde{g}_{3,49}$
$\tilde{g}_{3,50}$	$K_y + D + \epsilon P_z, P_y, P_t$	3	3	$\tilde{g}_{3,50}$
$\tilde{g}_{3,51}$	$K_y + D + \epsilon P_z, P_y, P_t + \alpha(P_x + P_z)$	3	3	$\tilde{g}_{3,51}$
$\tilde{g}_{3,52}$	$K_y + \alpha P_y + \beta P_t, P_x + P_z$	2	2	$g_{3,8}$
$\tilde{g}_{3,53}$	$K_y + \alpha P_y, P_x + P_z$	2	2	$g_{3,8}$
$\tilde{g}_{3,54}$	$K_y + \alpha P_t, P_x + P_z$	2	2	$g_{3,8}$
$\tilde{g}_{3,55}$	$K_y + \alpha P_t, P_y + \beta P_t$	2	2	$g_{3,17}$
$\tilde{g}_{3,56}$	$K_y + \alpha P_t, P_y$	2	2	$g_{3,17}$
$\tilde{g}_{3,57}$	$K_y + \alpha P_y, P_t$	2	2	$g_{3,17}$
$\tilde{g}_{3,58}$	$K_y + \alpha D, P_x + P_z$	2	2	$g_{3,18}$
$\tilde{g}_{3,59}$	$K_y + \alpha D, P_y$	2	2	$g_{3,19}$
$\tilde{g}_{3,60}$	$K_y + \alpha D, P_t$	2	2	$g_{3,20}$
$\tilde{g}_{3,61}$	$K_y + D, P_x + P_z + \alpha P_t$	2	2	$\tilde{g}_{3,61}$

$\tilde{g}_{3,62}$	$K_y + D + \epsilon P_z, P_x + P_z + \alpha P_t$	2	2	$\tilde{g}_{3,62}$
$\tilde{g}_{3,63}$	$K_y + D + \epsilon P_z, P_x + P_z$	2	2	$\tilde{g}_{3,63}$
$\tilde{g}_{3,64}$	$K_y + D + \epsilon P_z, P_y$	2	2	$\tilde{g}_{3,64}$
$\tilde{g}_{3,65}$	$K_y + D + \epsilon P_z, P_t$	2	2	$\tilde{g}_{3,65}$
$\tilde{g}_{3,66}$	$K_y + \alpha P_y + \beta P_t$	1	1	$g_{3,17}$
$\tilde{g}_{3,67}$	$K_y + \alpha P_y$	1	1	$g_{3,17}$
$\tilde{g}_{3,68}$	$K_y + \alpha P_t$	1	1	$g_{3,17}$
$\tilde{g}_{3,69}$	$K_y + \alpha D$	1	1	$g_{3,25}$
$\tilde{g}_{3,70}$	$K_y + D + \epsilon P_z$	1	1	$\tilde{g}_{3,70}$
$\tilde{g}_{4,18}$	$L_x + K_y + \alpha D, P_x, P_y, P_z, P_t$	5	5	$g_{4,0}$
$\tilde{g}_{4,19}$	$L_x + K_y + \alpha P_x, P_x + P_y, P_t + \beta P_y, P_z$	4	4	$g_{4,1}$
$\tilde{g}_{4,20}$	$L_x + K_y + \alpha P_t, P_x, P_y, P_z$	4	4	$\tilde{g}_{1,17} \|_{\alpha=1/2}$
$\tilde{g}_{4,21}$	$L_x + K_y + \alpha P_x, P_x + P_y, P_z, P_t$	4	4	$\tilde{g}_{4,21}$
$\tilde{g}_{4,22}$	$L_x + K_y + \alpha D, P_x, P_y, P_z$	4	4	$g_{4,2}$
$\tilde{g}_{4,23}$	$L_x + K_y + \alpha D, P_x + P_y, P_z, P_t$	4	4	$g_{4,3}$
$\tilde{g}_{4,24}$	$L_x + K_y + \alpha P_x, P_x + P_y, P_z + \beta P_t$	3	3	$\tilde{g}_{4,21}$
$\tilde{g}_{4,25}$	$L_x + K_y + \alpha P_x + \beta P_t, P_x + P_y, P_z$	3	3	$g_{4,1}$
$\tilde{g}_{4,26}$	$L_x + K_y + \alpha P_x, P_x + P_y, P_z$	3	3	$\tilde{g}_{1,17} \|_{\alpha=2}$
$\tilde{g}_{4,27}$	$L_x + K_y + \alpha P_t, P_x + P_y, P_z$	3	3	$\tilde{g}_{1,17} \|_{\alpha=1/2}$
$\tilde{g}_{4,28}$	$L_x + K_y + \alpha P_x, P_x + P_y, P_t$	3	3	$\tilde{g}_{1,23}$
$\tilde{g}_{4,29}$	$L_x + K_y + \alpha D, P_x + P_y, P_z$	3	3	$g_{4,7}$
$\tilde{g}_{4,30}$	$L_x + K_y + \alpha D, P_x + P_y, P_t$	3	3	$g_{4,8}$
$\tilde{g}_{4,31}$	$L_x + K_y + \alpha P_x + \beta P_t, P_x + P_y$	2	2	$\tilde{g}_{4,21}$
$\tilde{g}_{4,32}$	$L_x + K_y + \alpha P_x, P_x + P_y$	2	2	$\tilde{g}_{1,23}$
$\tilde{g}_{4,33}$	$L_x + K_y + \alpha P_t, P_x + P_y$	2	2	$\tilde{g}_{1,22} \|_{\alpha=1/2}$
$\tilde{g}_{4,34}$	$L_x + K_y + \alpha P_x, P_t$	2	2	$\tilde{g}_{1,31}$
$\tilde{g}_{4,35}$	$L_x + K_y + \alpha D, P_x + P_y$	2	2	$g_{4,12}$

$\tilde{g}_{4,36}$	$L_x + K_y + \alpha D, P_t$	2	2	$g_{4,13}$
$\tilde{g}_{4,37}$	$L_x + K_y + \alpha D + \epsilon P_z, P_t$	2	2	$\tilde{g}_{4,37}$
$\tilde{g}_{4,38}$	$L_x + K_y + \alpha P_x + \beta P_t$	1	1	$\tilde{g}_{4,28}$
$\tilde{g}_{4,39}$	$L_x + K_y + \alpha P_x$	1	1	$\tilde{g}_{4,28}$
$\tilde{g}_{4,40}$	$L_x + K_y + \alpha P_t$	1	1	$\tilde{g}_{1,29}\,\vert_{\alpha=1/2}$
$\tilde{g}_{4,41}$	$L_x + K_y + \alpha D$	1	1	$g_{4,16}$
$\tilde{g}_{4,42}$	$L_x + K_y + \alpha D + \epsilon P_z$	1	1	$\tilde{g}_{4,42}$

The main difference between algebras corresponding to the previous Eq. (1.27) and those corresponding to the present Eq. (3.1) is that here the Levi-Malcev decomposition is trivial. There is no semisimple part when generators L_y, L_z are substituted by generators K_y, K_z. The present algebra is solvable as was pointed out by J.Patera et al [51]. A large part of the analysis of the structure of subalgebras was presented in this paper. However, these authors considered only the 7-dimensional algebra which differs from the present one by the time translation generator. Patera et al called the corresponding group *the Similitude Group.*

c) The geometry of symmetry variables obtained through reduction

For the algebra corresponding to the case where at least two coefficients a, b, c, d are not equal to 0 one finds, as it was for the previous Eq. (3.1) the singular time-independent and spatially homogeneous solutions with values which are given by the roots of the polynomial (3.2). There are also time-dependent spatially homogeneous solutions satisfying the same equation as in the previous chapter.

The symmetry variables corresponding to standing planar waves and travelling planar waves are this time more complicated. As it can be easily seen from the form of Eq. (3.1) the direction x is distinguished from any other. Due to this fact equations describing the planar wave type solutions have coefficients of different signs.

The cylindrical symmetry variables appear also in this case but the axis of symmetry here must be parallel to the x axis.

The new variables include hyperbolic sheets, hyperboloids and some other more complicated variables involving combinations of exponential and algebraic functions. Surfaces of the constant value for some of the symmetry variables are sketched in Fig. 3.4.

d) Solutions of algebraic and first order equations from Tables 3.8 - 3.10

The ordinary differential equations obtained through the symmetry reduction method for Eq. (3.24) are listed in Tables 3.8 - 3.10. Among them there are many first order equations and an algebraic one.

$$a + bf + cf^3 + df^5 = 0 \tag{3.25}$$

$$\frac{df}{d\xi} = a + bf + cf^3 + df^5 \tag{3.26}$$

For the above equations there are versions corresponding to cases $a = b = c = 0$ and $a = b = d = 0$ while equations listed below appear only in the one of these cases.

$$(\alpha - 1)\frac{df}{d\xi} - \frac{1}{2}f = cf^3 \tag{3.27}$$

$$(\alpha - 1)\frac{df}{d\xi} - \frac{1}{4}f = df^5 \tag{3.28}$$

$$\alpha\frac{df}{d\xi} - \frac{1}{2}f = cf^3 \tag{3.29}$$

$$\alpha\frac{df}{d\xi} - \frac{1}{4}f = df^5 \tag{3.30}$$

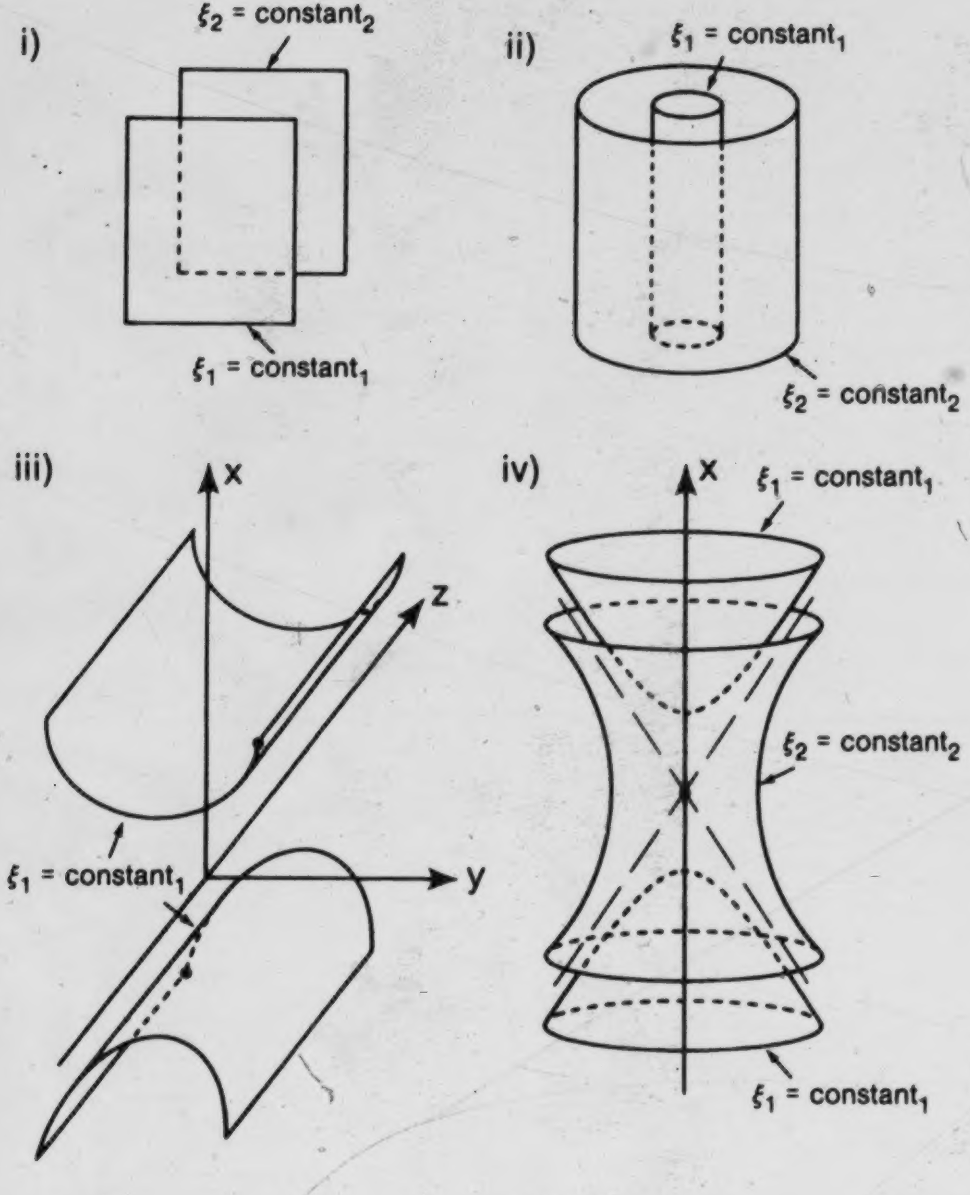
i)
ξ_2 = constant$_2$
ξ_1 = constant$_1$
ii)
ξ_1 = constant$_1$
ξ_2 = constant$_2$
iii)
x
z
y
ξ_1 = constant$_1$
iv)
x
ξ_1 = constant$_1$
ξ_2 = constant$_2$
ξ_1 = constant$_1$

Figure 3.4: Geometries of some symmetry variables obtained by reduction of the Eq. $\partial_t M + \partial_{xx} M - \partial_{yy} M - \partial_{zz} M = \gamma M^q$ for $q = 3, 5$. i) planar waves; ii) cylindrical waves around the x axis; iii) parabolical sheets; iv) hyperbolical surfaces.

Table 3.8: Reductions of the equation $\partial_t\eta + \partial_{xx}\eta - \partial_{yy}\eta - \partial_{zz}\eta = a + b\eta + c\eta^3 + d\eta^5$ to the ordinary differential equations where at least two of the coefficients a, b, c, d one of which is c or d are different from 0; $\eta(x, y, z, t) = f(\xi)$ where ξ is a function of x, y, z, t; α and β are different from 0.

No	ξ	Reduced Equation	Painlevé Test
1	t	$\frac{df}{d\xi} = a + bf + cf^3 + df^5$	1^{st}-order
2	x	$\frac{d^2f}{d\xi^2} = a + bf + cf^3 + df^5$	yes for $a = 0$ or $d = 0$
3	y	$\frac{d^2f}{d\xi^2} = -a - bf - cf^3 - df^5$	yes for $a = 0$ or $d = 0$
4	$x + \alpha t$	$\frac{d^2f}{d\xi^2} + \alpha\frac{df}{d\xi} = a + bf + cf^3 + df^5$	yes for $[a = c = 0$ and $b = -\frac{3\alpha^2}{16}]$
			or $[a = d = 0$ and $b = -\frac{2\alpha^2}{9}]$
5	$y - \alpha t$	$\frac{d^2f}{d\xi^2} + \alpha\frac{df}{d\xi} = -a - bf - cf^3 - df^5$	yes for $[a = c = 0$ and $b = \frac{3\alpha^2}{16}]$
			or $[a = d = 0$ and $b = \frac{2\alpha^2}{9}]$
6	$x - y$	$0 = a + bf + cf^3 + df^5$	algebraic equation
7	$t - \alpha(x - y)$	$\frac{df}{d\xi} = a + bf + cf^3 + df^5$	1^{st}-order
8	$\sqrt{x^2 - y^2}$	$\frac{d^2f}{d\xi^2} + \frac{1}{\xi}\frac{df}{d\xi} = a + bf + cf^3 + df^5$	no
9	$\sqrt{y^2 + z^2}$	$\frac{d^2f}{d\xi^2} + \frac{1}{\xi}\frac{df}{d\xi} = -a - bf - cf^3 - df^5$	no
10	$\sqrt{x^2 - y^2 - z^2}$	$\frac{d^2f}{d\xi^2} + \frac{2}{\xi}\frac{df}{d\xi} = a + bf + cf^3 + df^5$	no
11	$\ln\|x - z\| + \frac{t}{\alpha}$	$\frac{df}{d\xi} = \alpha(a + bf + cf^3 + df^5)$	1^{st}-order
12	$\ln\|x - z\| + \frac{y}{\alpha}$	$\frac{d^2f}{d\xi^2} = \alpha^2(-a - bf - cf^3 - df^5)$	yes for $[a = 0$ or $d = 0]$

13	$\ln\lvert x-z\rvert+\frac{t}{\alpha}-y$	$\frac{d^2f}{d\xi^2}-\frac{1}{\alpha}\frac{df}{d\xi}=-a-bf-cf^3-df^5$	yes for $[a=c=0$ and $b=\frac{3}{16\alpha^2}]$ or $[a=d=0$ and $b=\frac{2}{9\alpha^2}]$
14	$2\alpha z-(x-y)^2$	$\frac{d^2f}{d\xi^2}=\frac{1}{4\alpha^2}(-a-bf-cf^3-df^5)$	yes for $a=0$ or $d=0$
15	$(z-\alpha t)+\frac{(x-y)^2}{2\beta}$	$\frac{d^2f}{d\xi^2}+\alpha\frac{df}{d\xi}=-a-bf-cf^3-df^5$	yes for $[a=c=0$ and $b=-\frac{3\alpha^2}{16}]$ or $[a=d=0$ and $b=-\frac{2\alpha^2}{9}]$

Table 3.9: Reductions of the equation $\partial_t\eta + \partial_{xx}\eta - \partial_{yy}\eta - \partial_{zz}\eta = c\eta^3$ to ordinary differential equations; $\eta(x,y,z,t) = \rho f(\xi)$ where ρ and ξ are functions of x, y, z, t; α and β are different from 0.

No	ξ	ρ	Reduced Equation	Painlevé Test
1	t	1	$\frac{df}{d\xi} = cf^3$	1st order
2	x	1	$\frac{d^2f}{d\xi^2} = cf^3$	yes
3	y	1	$\frac{d^2f}{d\xi^2} = -cf^3$	yes
4	$x - \alpha t$	1	$\frac{d^2f}{d\xi^2} - \alpha\frac{df}{d\xi} = cf^3$	no
5	$y - \alpha t$	1	$\frac{d^2f}{d\xi^2} + \alpha\frac{df}{d\xi} = -cf^3$	no
6	$x - y$	1	$0 = cf^3$	algebraic
7	$t - \alpha(x - y)$	1	$\frac{df}{d\xi} = cf^3$	1st order
8	$\sqrt{x^2 - y^2}$	1	$\frac{d^2f}{d\xi^2} + \frac{1}{\xi}\frac{df}{d\xi} = f^3$	no
9	$\sqrt{y^2 + z^2}$	1	$\frac{d^2f}{d\xi^2} + \frac{1}{\xi}\frac{df}{d\xi} = -cf^3$	no
10	$\sqrt{x^2 - y^2 - z^2}$	1	$\frac{d^2f}{d\xi^2} + \frac{2}{\xi}\frac{df}{d\xi} = cf^3$	no
11	$\ln\|x - z\| + \frac{t}{\alpha}$	1	$\frac{df}{d\xi} = \alpha cf^3$	1st order
12	$\ln\|x - z\| + \frac{y}{\alpha}$	1	$\frac{d^2f}{d\xi^2} = -c\alpha^2 f^3$	yes
13	$\ln\|x - z\| + \frac{t}{\alpha} - y$	1	$\frac{d^2f}{d\xi^2} - \frac{1}{\alpha}\frac{df}{d\xi} = -cf^3$	no
14	$2\alpha z - (x - y)^2$	1	$\frac{d^2f}{d\xi^2} = \frac{-c}{4\alpha^2}f^3$	yes
15	$(z - \alpha t) + \frac{(x-y)^2}{2\beta}$	1	$\frac{d^2f}{d\xi^2} + 2\alpha\frac{df}{d\xi} = -cf^3$	no

16	$\frac{t}{x^2-y^2-z^2}$	$(x^2-y^2-z^2)^{-1/2}$	$4\xi^2\frac{d^2f}{d\xi^2}+(6\xi+1)\frac{df}{d\xi}=cf^3$	no
17	$\frac{t}{x^2}$	x^{-1}	$4\xi^2\frac{d^2f}{d\xi^2}+(10\xi+1)\frac{df}{d\xi}+2f=cf^3$	no
18	$\frac{t}{y^2}$	y^{-1}	$4\xi^2\frac{d^2f}{d\xi^2}+(10\xi-1)\frac{df}{d\xi}+2f=-cf^3$	no
19	$\frac{t}{y^2+z^2}$	$(y^2+z^2)^{-1/2}$	$4\xi^2\frac{d^2f}{d\xi^2}+(8\xi-1)\frac{df}{d\xi}+f=-cf^3$	no
20	$\frac{x^2}{y^2+z^2}$	$(y^2+z^2)^{-1/2}$	$4\xi(\xi-1)\frac{d^2f}{d\xi^2}+(8\xi-2)\frac{df}{d\xi}+f=-cf^3$	no
21	$\frac{t}{x^2-y^2}$	$(x^2-y^2)^{-1/2}$	$4\xi^2\frac{d^2f}{d\xi^2}+(8\xi+1)\frac{df}{d\xi}+f=cf^3$	no
22	$\frac{y^2}{x^2-z^2}$	$(x^2-z^2)^{-1/2}$	$4\xi(\xi-1)\frac{d^2f}{d\xi^2}+(8\xi-2)\frac{df}{d\xi}+f=cf^3$	no
23	$\frac{t}{(x-y)^2}$	$(x-y)^{-1}$	$\frac{df}{d\xi}=cf^3$	1^{st} order
24	$\frac{(x-y)^2}{x^2-y^2-z^2}$	$(x^2-y^2-z^2)^{-1/2}$	$4\xi^2\frac{d^2f}{d\xi^2}+6\xi\frac{df}{d\xi}=-cf^3$	no
25	$\arctan(\frac{y}{z})$	$(y^2+z^2)^{-1/2}$	$\frac{d^2f}{d\xi^2}+f=-cf^3$	yes
26	$\frac{x}{y}$	y^{-1}	$(\xi^2-1)\frac{d^2f}{d\xi^2}+4\xi\frac{df}{d\xi}+2f=-cf^3$	yes
27	$\ln\lvert x-y\rvert-\ln\lvert z\rvert$	z^{-1}	$\frac{d^2f}{d\xi^2}+3\frac{df}{d\xi}+2f=-cf^3$	yes
28	$-\frac{2\alpha}{\alpha^2+1}(\arctan\frac{y}{z}+\frac{1}{2}\alpha\log(y^2+z^2))$	$(\frac{4\alpha^2}{\lvert c\rvert(\alpha^2+1)(y^2+z^2)})^{1/2}$	$\frac{d^2f}{d\xi^2}+\frac{df}{d\xi}+\frac{\alpha^2+1}{4\alpha^2}f+sgn(c)f^3=0$	yes for $\alpha=\pm 3i$
29	$(\alpha-1)-2\ln(x-z)$	$t^{-1/2}$	$(\alpha-1)\frac{df}{d\xi}-\frac{1}{2}f=cf^3$	1^{st} order
30	$\frac{1}{2}\ln\lvert\frac{x-y}{x+y}\rvert$	$(x^2-y^2)^{-1/2}$	$\frac{d^2f}{d\xi^2}-f=-cf^3$	yes
31	$x-y$	$(t-\alpha\frac{x+y}{2})^{-1/2}$	$\alpha\frac{df}{d\xi}-\frac{1}{2}f=cf^3$	1^{st} order
32	$\frac{t+\alpha(x-z)}{y^2}$	y^{-1}	$4\xi^2\frac{d^2f}{d\xi^2}+(10\xi-1)\frac{df}{d\xi}+2f=-cf^3$	no
33	$\frac{\alpha z}{x-y}-\ln\lvert x-y\rvert$	$(x-y)^{-1}$	$\frac{d^2f}{d\xi^2}=-\frac{c}{\alpha^2}f^3$	yes
34	$2\alpha(x-y)-(\alpha-1)\ln t$	$t^{-1/2}$	$(1-\alpha)\frac{df}{d\xi}-\frac{1}{2}f=cf^3$	1^{st} order

35	$(x-z)(z-(x-y)\frac{t}{\alpha})$	$x-y$	$\cancel{\frac{d^2f}{d\xi^2}+}\frac{1}{\alpha}\frac{df}{d\xi} = -cf^3$	no
36	$\log(\frac{(x^2-y^2-z^2)^{\alpha-1}}{(x-y)^{2\alpha}})$	$(x-y)^{\frac{\alpha}{1-\alpha}}$	$\frac{d^2f}{d\xi^2} - \frac{1}{2(1-\alpha)}\frac{df}{d\xi} = \frac{c}{4(1-\alpha^2)}\exp(\frac{\xi}{\alpha-1})f^3$	do not know
37	$x-y$	$(\frac{2(x-y)}{x^2-y^2-z^2})^{1/2}$	$\frac{df}{d\xi} + \frac{1}{2\xi}f = -cf^3$	1st order
38	$\frac{(x+y+\frac{(x-y)^3}{6\alpha^2}-z\frac{x-y}{\alpha})^2}{(z-\frac{(x-y)^2}{4\alpha})^3}$	$(z-\frac{(x-y)^2}{4\alpha})^{-1}$	$\xi(9\xi+\frac{16}{\alpha})\frac{d^2f}{d\xi^2} + (18\xi-\frac{8}{\alpha})\frac{df}{d\xi} + 2f = -cf^3$	no
39	$x-y$	$(x+y-\frac{2t}{\alpha}-\frac{z^2}{x-y})^{-1/2}$	$2\frac{df}{d\xi} + f(\frac{1}{\xi}-\frac{1}{\alpha}) = -cf^3$	1st order
40	$\ln\lvert y\rvert + x - z$	y^{-1}	$\frac{d^2f}{d\xi^2} - 3\frac{df}{d\xi} + 2f = -cf^3$	yes
41	$2(x-z)+\ln t$	$t^{-1/2}$	$\frac{df}{d\xi} - \frac{1}{2}f = cf^3$	1st order
42	$(x+z+\frac{1}{2})\exp 2(x-z)$	$(x+z+\frac{1}{2})^{-1/2}$	$8\xi^2\frac{d^2f}{d\xi^2} + 4\xi\frac{df}{d\xi} = cf^3$	no
43	$\ln\lvert x+z-2\alpha t+\frac{1}{2}\rvert + 2(x-z)$	$(x+z-2\alpha t+\frac{1}{2})^{-1/2}$	$\frac{d^2f}{d\xi^2} - \frac{2+\alpha}{4}\frac{df}{d\xi} + \frac{\alpha}{8}f = \frac{c}{8}f^3$	yes for $\alpha = -2, 1, 4$
44	$t^{-1/2}(z-\frac{(x-y)^2}{2\alpha})$	$t^{-1/2}$	$\frac{d^2f}{d\xi^2} + \frac{1}{2}\frac{df}{d\xi} + \frac{1}{2}f = -cf^3$	no
45	$x-y$	$[\frac{1}{2}(x+y-\frac{z^2}{x-y+\alpha})]^{-1/2}$	$\frac{df}{d\xi} + \frac{1}{2(\xi+\alpha)}f = -cf^3$	1st order
46	$\ln\lvert x-y\rvert + \frac{1}{2}\ln t$	$t^{-1/2}$	$\frac{df}{d\xi} + f = 2cf^3$	1st order
47	$\ln\lvert(x-y-\frac{1}{2})t\rvert$	$t^{-1/2}$	$\frac{df}{d\xi} - \frac{1}{2}f = cf^3$	1st order
48	$\ln\lvert x-y+\frac{1}{2}\rvert - \ln t$	$t^{-1/2}$	$\frac{df}{d\xi} + \frac{1}{2}f = -cf^3$	1st order
49	$\frac{x+y}{2} - \frac{z^2}{2(x-y+1/2)} - \frac{1}{4} + \frac{1}{4}\ln(x-y+\frac{1}{2})$	$(x-y+1/2)^{-1/2}$	$\frac{d^2f}{d\xi^2} = 2cf^3$	yes

Table 3.10: Reduction of the equation $\partial_t\eta+\partial_{xx}\eta-\partial_{yy}\eta-\partial_{zz}\eta = d\eta^5$ to ordinary differential equations; $\eta(x,y,z,t) = \rho f(\xi)$ where ρ and ξ are functions of x,y,z,t; α,β are different from 0.

No	ξ	ρ	Reduced Equation	Painlevé Test
1	t	1	$\frac{df}{d\xi} = df^5$	1st order
2	x	1	$\frac{d^2f}{d\xi^2} = df^5$	yes
3	y	1	$\frac{d^2f}{d\xi^2} = -df^5$	yes
4	$x-\alpha t$	1	$\frac{d^2f}{d\xi^2} - \alpha\frac{df}{d\xi} = df^5$	no
5	$y-\alpha t$	1	$\frac{d^2f}{d\xi^2} + \alpha\frac{df}{d\xi} = -df^5$	no
6	$x-y$	1	$0 = df^5$	algebraic
7	$t-\alpha(x-y)$	1	$\frac{df}{d\xi} = df^5$	1st order
8	$\sqrt{x^2-y^2}$	1	$\frac{d^2f}{d\xi^2} + \frac{1}{\xi}\frac{df}{d\xi} = df^5$	no
9	$\sqrt{y^2+z^2}$	1	$\frac{d^2f}{d\xi^2} + \frac{1}{\xi}\frac{df}{d\xi} = -df^5$	no
10	$\sqrt{x^2-y^2-z^2}$	1	$\frac{d^2f}{d\xi^2} + \frac{2}{\xi}\frac{df}{d\xi} = df^5$	yes
11	$\ln\lvert x-z\rvert + \frac{t}{\alpha}$	1	$\frac{df}{d\xi} = \alpha df^5$	1st order
12	$\ln\lvert x-z\rvert + \frac{y}{\alpha}$	1	$\frac{d^2f}{d\xi^2} = -\alpha^2 df^5$	yes
13	$\ln\lvert x-z\rvert + \frac{t}{\alpha} - y$	1	$\frac{d^2f}{d\xi^2} - \frac{1}{\alpha}\frac{df}{d\xi} = -df^5$	no
14	$2\alpha z-(x-y)^2$	1	$\frac{d^2f}{d\xi^2} = \frac{-d}{4\alpha^2}f^5$	yes
15	$(z-\alpha t)+\frac{(x-y)^2}{2\beta}$	1	$\frac{d^2f}{d\xi^2} + \alpha\frac{df}{d\xi} = -df^5$	no

16	$\frac{t}{x^2-y^2-z^2}$	$(x^2-y^2-z^2)^{-1/4}$	$4\xi^2\frac{d^2f}{d\xi^2}+(4\xi+1)\frac{df}{d\xi}=df^5$	no
17	$\frac{t}{x^2}$	$x^{-1/2}$	$4\xi^2\frac{d^2f}{d\xi^2}+(8\xi+1)\frac{df}{d\xi}+\frac{3}{4}f=df^5$	no
18	$\frac{t}{y^2}$	$y^{-1/2}$	$4\xi^2\frac{d^2f}{d\xi^2}+(8\xi-1)\frac{df}{d\xi}+\frac{3}{4}f=-df^5$	no
19	$\frac{t}{y^2+z^2}$	$(y^2+z^2)^{-1/4}$	$4\xi^2\frac{d^2f}{d\xi^2}+(6\xi-1)\frac{df}{d\xi}+\frac{1}{4}f=-df^5$	no
20	$\frac{z^2}{y^2+z^2}$	$(y^2+z^2)^{-1/4}$	$4\xi(\xi-1)\frac{d^2f}{d\xi^2}+(6\xi-2)\frac{df}{d\xi}+\frac{1}{4}f=-df^5$	no
21	$\frac{t}{x^2-y^2}$	$(x^2-y^2)^{-1/4}$	$4\xi^2\frac{d^2f}{d\xi^2}+(6\xi+1)\frac{df}{d\xi}+\frac{1}{4}f=df^5$	no
22	$\frac{y^2}{x^2-z^2}$	$(x^2-z^2)^{-1/4}$	$4\xi(\xi-1)\frac{d^2f}{d\xi^2}+(6\xi-2)\frac{df}{d\xi}+\frac{1}{4}f=df^5$	no
23	$\frac{t}{(x-y)^2}$	$(x-y)^{-1/2}$	$\frac{df}{d\xi}=df^5$	1st order
24	$\frac{(x-y)^2}{x^2-y^2-z^2}$	$(x^2-y^2-z^2)^{-1/4}$	$4\xi^2\frac{d^2f}{d\xi^2}+6\xi\frac{df}{d\xi}+\frac{1}{4}f=-df^5$	no
25	$\arctan(\frac{y}{z})$	$(y^2+z^2)^{-1/2}$	$\frac{d^2f}{d\xi^2}+\frac{1}{4}f=-df^5$	yes
26	$\frac{x}{y}$	$y^{-1/2}$	$(\xi^2-1)\frac{d^2f}{d\xi^2}+3\xi\frac{df}{d\xi}+\frac{3}{4}f=-df^5$	yes
27	$\ln\|x-y\|-\ln\|z\|$	$z^{-1/2}$	$\frac{d^2f}{d\xi^2}+2\frac{df}{d\xi}+\frac{3}{4}f=-df^5$	yes
28	$-\frac{2\alpha}{\alpha^2+1}(\arctan\frac{y}{z}+\frac{1}{2}\alpha\log(y^2+z^2))$	$(\frac{4\alpha^2}{\|d\|(\alpha^2+1)(y^2+z^2)})^{1/4}$	$\frac{d^2f}{d\xi^2}+\frac{df}{d\xi}+\frac{\alpha^2+1}{4\alpha^2}f+sgn(d)f^5=0$	yes for $\alpha=\pm 2i$
29	$(\alpha-1)\ln t-2\ln\|x-z\|$	$t^{-1/4}$	$(\alpha-1)\frac{df}{d\xi}-\frac{1}{4}f=df^5$	1st order
30	$\frac{1}{2}\ln\|\frac{x-y}{x+y}\|$	$(x^2-y^2)^{-1/4}$	$\frac{d^2f}{d\xi^2}-\frac{1}{4}f=-df^5$	yes
31	$x-y$	$(t-\alpha\frac{x+y}{2})^{-1/4}$	$\alpha\frac{df}{d\xi}-f=2df^5$	1st order
32	$\frac{t+\alpha(x-z)}{y^2}$	$y^{-1/2}$	$4\xi^2\frac{d^2f}{d\xi^2}+(8\xi-1)\frac{df}{d\xi}+\frac{3}{4}f=-df^5$	no
33	$\frac{\alpha z}{x-y}-\ln\|x-y\|$	$(x-y)^{-1/2}$	$\frac{d^2f}{d\xi^2}=\frac{-d}{\alpha^2}f^5$	yes
34	$\frac{2}{\alpha}\ln\|x-y\|-(\alpha-1)\ln t$	$t^{-1/4}$	$(1-\alpha)\frac{df}{d\xi}-\frac{1}{4}f=df^5$	1st order

35	$(x-z)(z-(x-y)\frac{t}{\alpha})$	$(x-y)^{1/2}$	$\frac{d^2f}{d\xi^2}+\frac{1}{\alpha}\frac{df}{d\xi}=-df^5$	no
36	$\log(\frac{(x^2-y^2-z^2)^{\alpha-1}}{(x-y)^{2\alpha}})$	$(x-y)^{\frac{\alpha}{2(1-\alpha)}}$	$\frac{d^2f}{d\xi^2}-\frac{1}{2(1-\alpha^2)}\frac{df}{d\xi}=\frac{d}{4(1-\alpha^2)}\exp(\frac{\xi}{\alpha-1})f^5$	do not know
37	$x-y$	$(\frac{2(x-y)}{x^2-y^2-z^2})^{1/4}$	$\frac{df}{d\xi}+\frac{1}{2\xi}f=-df^5$	1st order
38	$\frac{(x+y+\frac{(x-y)^3}{6\alpha^2}-z\frac{x-y}{\alpha})^2}{(z-\frac{(x-y)^2}{4\alpha})^3}$	$(z-\frac{(x-y)^2}{4\alpha^2})^{-1/2}$	$\xi(9\xi+\frac{16}{\alpha})\frac{d^2f}{d\xi^2}+(15\xi-\frac{8}{\alpha})\frac{df}{d\xi}+\frac{3}{4}f=-df^5$	no
39	$x-y$	$(x+y-\frac{2t}{\alpha}-\frac{z^2}{x-y})^{-1/4}$	$\frac{df}{d\xi}+\frac{1}{2}f(\frac{1}{\xi}-\frac{1}{\alpha})=-df^5$	1st order
40	$\ln\|y\|+x-z$	$y^{-1/2}$	$\frac{d^2f}{d\xi^2}-2\frac{df}{d\xi}+\frac{3}{4}f=-df^5$	yes
41	$\ln t+2(x-z)$	$t^{-1/4}$	$\frac{df}{d\xi}-\frac{1}{4}f=df^5$	1st order
42	$(x+z+\frac{1}{2})\exp 2(x-z)$	$(x+z+\frac{1}{2})^{-1/4}$	$8\xi^2\frac{d^2f}{d\xi^2}+6\xi\frac{df}{d\xi}=df^5$	no
43	$\ln\|x+z-2\alpha t+\frac{1}{2}\|+2(x-z)$	$(x+z-2\alpha t+\frac{1}{2})^{-1/2}$	$\frac{d^2f}{d\xi^2}-\frac{1+\alpha}{4}\frac{df}{d\xi}+\frac{\alpha}{16}f=\frac{d}{8}f^5$	yes for $\alpha=-1,1,3$
44	$t^{-1/2}(z-\frac{(x-y)^2}{2\alpha})$	$t^{-1/4}$	$\frac{d^2f}{d\xi^2}+\frac{1}{2}\frac{df}{d\xi}+\frac{1}{4}f=-df^5$	no
45	$x-y$	$(\frac{1}{2}(x+y-\frac{z^2}{x-y+\alpha}))^{-1/4}$	$\frac{df}{d\xi}+\frac{1}{2(\xi+\alpha)}f=-2df^5$	1st order
46	$\ln\|x-y\|+\frac{1}{2}\ln t$	$t^{-1/4}$	$\frac{df}{d\xi}-\frac{1}{2}f=2df^5$	1st order
47	$\ln\|(x-y-\frac{1}{2})t\|$	$t^{-1/4}$	$\frac{df}{d\xi}-\frac{1}{4}f=df^5$	1st order
48	$\ln\|x-y+\frac{1}{2}\|-\ln t$	$t^{-1/4}$	$\frac{df}{d\xi}+\frac{1}{4}f=-df^5$	1st order
49	$\frac{x+y}{2}-\frac{z^2}{2(x-y+1/2)}-\frac{1}{4}+\frac{1}{4}\ln(x-y+\frac{1}{2})$	$(x-y+1/2)^{-1/4}$	$\frac{d^2f}{d\xi^2}+\frac{df}{d\xi}=2df^5$	no

$$(1-\alpha)\frac{df}{d\xi}-\frac{1}{2}f=cf^3 \tag{3.31}$$

$$(1-\alpha)\frac{df}{d\xi}-\frac{1}{4}f=df^5 \tag{3.32}$$

$$\frac{df}{d\xi}+\frac{1}{2\xi}f=-cf^3 \tag{3.33}$$

$$\frac{df}{d\xi}+\frac{1}{2\xi}f=-df^5 \tag{3.34}$$

$$2\frac{df}{d\xi}+(\frac{1}{\xi}-\frac{1}{\alpha})f=-cf^3 \tag{3.35}$$

$$\frac{df}{d\xi}+\frac{1}{2}(\frac{1}{\xi}-\frac{1}{\alpha})f=-df^5 \tag{3.36}$$

$$\frac{df}{d\xi}+\frac{1}{2(\xi+\alpha)}f=-cf^3 \tag{3.37}$$

$$\frac{df}{d\xi}+\frac{1}{2(\xi+\alpha)}f=-2df^5 \tag{3.38}$$

Only one algebraic and one first order reduction existed for the Eq. (3.1). The algebraic one was related to solutions having the full symmetry. The first order equation corresponded to the spatially-homogeneous time-dependent solution. One may expect, therefore, a similar behaviour for the present Eq. (3.24). However, there are many new equations in this case.

To explain these phenomena one has to realize that there is a variable of a quite special type now. The symmetry variable of the form $\xi = x \pm y$ (or equivalently any conjugated variable) is degenerate in the sense that the differential operator in Eq. (3.24) vanishes on the arbitrary function of ξ. Therefore, any other symmetry variable of the form $\xi(x \pm y, t)$ will lead, after reduction, to a first order ODE. It also may happen that a combination of variables $x \pm y$ and t appears as a function ρ in the reduction.

Solutions of Eqs. (3.25,3.26) were discussed in the previous chapter (3.2,3.3). Methods of solving and an interpretation of results were also given there.

The remaining equations can be divided into two classes with respect to the coefficient of the first derivative term and to the dependence on ξ. The method of solution is given for each class separately with an initial condition $y_0 = y(x_0)$.

The first class contains equations that do not contain the independent variable ξ explicitly. These are Eqs. (3.27,3.28,3.29,3.30,3.31,3.32). These equations are of the Bernoulli type [28, page 19]

$$y' + f(x)y + g(x)y^{\beta} = 0 \tag{3.39}$$

and after the substitution $u(x) = y^{1-\beta}$ are transformed into a linear ODE of the form

$$u' + (1-\beta)f(x)u + (1-\beta)g(x) = 0.$$

The general solution of Eq. (3.39) with the initial condition (x_0, y_0) is in the form

$$y(x) = [\exp(-F)(y_0^{1-\beta} - \int_{x_0}^{x}(1-\beta)g(x)\exp(F))dx]^{\frac{1}{1-\beta}} \tag{3.40}$$

where

$$F(x) = \int_{x_0}^{x}(1-\beta)f(x)dx.$$

For the above listed cases the functions f and g are constant. Denoting them by f and g, respectively, one obtains the general solution in the form

$$y(x) = [\exp(-f(x-x_0))y_0^{1-\beta} - (1-\beta)g]^{\frac{1}{1-\beta}}$$

The second class of equations contains the ξ^{-1} coefficient by the linear term in these equations. These equations are: (3.33,3.34,3.35,3.36,3.37,3.38). These equations are also of the Bernoulli type (3.39). However, the function f is different from 0 here. Using the general form of the solution one finds the following explicit solutions: to Eq. (3.33)

$$y(\xi) = [\frac{x}{x_0}(y_0^{-2} - \frac{c}{x_0}(x^2 - x_0^2))]^{-1/2}$$

to Eq. (3.34)

$$y(\xi) = [(\frac{x}{x_0})^2(y_0^{-4} - 4dx_0^2(x^{-1} - x_0^{-1}))]^{-1/4}$$

to Eq. (3.35)

$$y(\xi) = \sqrt{\frac{x_0}{x}}[\exp(-\frac{x - x_0}{\alpha})(y_0^{-2} + \alpha c - \frac{\alpha^2 c}{x_0}) - \alpha c + \frac{\alpha^2 c}{x_0}]^{-1/2}$$

to Eq. (3.36)

$$y(\xi) = \sqrt{\frac{x_0}{x}}[\exp(-\frac{2(x - x_0)}{\alpha})(y_0^{-4} + 4d + \frac{\alpha}{2}(Ei(\frac{2x}{\alpha}) - Ei(\frac{2x_0}{\alpha})) - 4d\frac{x_0}{x}]^{-1/4},$$

where Ei is an exponential integral defined in [21, page xxxii],

to Eq. (3.37)

$$y(\xi) = [\frac{x + \alpha}{x_0 + \alpha}(y_0^{-2} - \frac{2c\alpha}{x_0 + \alpha}(x - x_0) - \frac{c}{x_0 + \alpha}(x^2 - x_0^2))]^{-1/2}$$

and to Eq. (3.38)

$$y(\xi) = [(\frac{x + \alpha}{x_0 + \alpha})^2(y_0^{-4} + 8d(x + \alpha)(x^{-1} - x_0^{-1}))]^{-1/2}.$$

e) Equations satisfying the Painlevé test

There are many equations in Tables 3.8 - 3.10 that satisfy the Painlevé test. Most of them are the same as those from the previous chapter and they were discussed already. In order not to repeat the discussion only the following new second order equations that satisfy the Painlevé test are listed below:

$$(\xi^2 - 1)\frac{d^2 f}{d\xi^2} + 4\xi\frac{df}{d\xi} + 2f = -cf^3 \tag{3.41}$$

$$(\xi^2 - 1)\frac{d^2 f}{d\xi^2} + 3\xi\frac{df}{d\xi} + \frac{3}{4}f = -df^5 \tag{3.42}$$

Solutions of equation (3.41)

Equation (3.41) can be solved after it is rewritten in the form

$$\frac{d^2}{d\xi^2}[(\xi^2 - 1)f] = -cf^3. \tag{3.43}$$

The next substitution $f(\xi) = (\xi^2 - 1)^{(-1/2)}w(\xi)$ transforms Eq. (3.43) into

$$(\xi^2 - 1)^2\frac{d^2w}{d\xi^2} + 2\xi(\xi^2 - 1)\frac{dw}{d\xi} = -w - cw^3.$$

This equation is in the form described by E.Kamke in his book [28, p.569] as equation 6.103. The next substitution changes the independent variable $x = \ln|(\xi - 1)/(\xi + 1)|$ so that this equation is reduced to the form

$$w''(x) = -w(x) - cw^3(x). \tag{3.44}$$

The last equation can be once integrated and Eq. (3.14) appears once more. Therefore, the final solution of Eq. (3.41) is of the form

$$f(\xi) = \frac{w(\ln|(\xi - 1)/(\xi + 1)|)}{\sqrt{\xi^2 - 1}}$$

where w is a solution of Eq. (3.44) which is expressed in terms of Jacobi elliptic functions.

Solutions of equation (3.42)

Equation (3.42) can be solved using substitutions written down by Winternitz et al [67]. The final solution is in the form:

$$f(\xi) = \sqrt{\sqrt{\frac{-3}{d(\xi^2 - 1)}}w(\pm\ln|\frac{\xi - 1}{\xi + 1}| + K)}$$

where the w function satisfies the Painlevé XXX equation

$$w'' = \frac{w'^2}{2w} + \frac{3}{2}w^3 + \frac{1}{2}w. \tag{3.45}$$

The Painlevé equation XXX (3.45) has the first integral in the Jacobi form and it was already discussed as Eq. (3.20).

3.3 The numerical values of the free energy functional for selected patterns

In this section calculations of numerical values of the free energy functional for some of the symmetry variables are presented.

To understand the obtained results more clearly it is convenient to reduce the number of the coefficients. Therefore, it is assumed that the coefficients $A_6 = A_0 = 0$. Analytical formulae for the free energy of the planar wave type solutions for this choice of the coefficients are available [66]. They can be used for comparison with numerical estimates.

There exist various methods of setting up the physical situation. They are expressed in terms of coefficients as well as boundary conditions. Our approach neglects boundary conditions in the sense that in the derivation of the fundamental equation (1.26) the surface terms were assumed to vanish. It is a limiting assumption because there is no reason to believe that the magnetization vector or its fluctuations should be equal to 0 on the boundary of the sample. However, the validity of these results can be extended to all those cases where the length of the sample is much greater than the period of the related oscillation. In such a case the relative error is small in comparison to the absolute value of the free energy functional.

A phase transition in the Landau theory is related to the change in the number of minima of the free energy functional signalling a bifurcation. It was achieved by assuming that in the vicinity of the transition point only the coefficient of the quadratic term is temperature dependent, or in other words, that the remaining coefficients are slowly varying and different from zero in this region.

It is possible, however, to discuss a different type of phase transition. The coefficient standing by the gradient term might change its sign in the vicinity of the transition point and the quartic term might remain constant. This possibility was discussed by J.A. Tuszyński et al [62].

On the microscopic level it is quite normal to assume that one of the coupling constants may change its sign. The results of a such an effect were for the first time explored by by C. Kittel [33] and later on were generalized to create the theory known as RKKY. This theory introduces the change of the sign of the exchange constant J in the Heisenberg hamiltonian as a result of the exchange constant's distance dependence.

Another attempt to describe the reentrant phase transitions was made by H. Kitatani et al [32]. They have assumed that in a two-dimensional lattice there is a competition between exchange coefficients changing signs at different lattice sites. In this way they have obtained some phase diagrams plotted in the space of the ratios of the exchange coefficients.

Here, according to our interpretation of the coarse grained free energy functional we propose that the interaction between planes in a sample may have different character than those within planes (anisotropy). The main question that appears now is: what kind of phase transitions might one expect in such a situation?

It is obvious that the change of the sign of one of the coefficients D_i may change the symmetry group. The crucial word here is "may" because despite the fact that the large (high-temperature) groups are quite different they have some common subgroups.

The analysis of the transition sequence as D in Eq. (1.26) goes through zero performed by J.A. Tuszyński et al [62] was limited to the planar wave type solutions. It was deduced that there are four possible sequences :

$A > 0$, $D > 0$ As $D \rightarrow 0$ $M = 0$ remains the only solution

$A < 0$, $D > 0$ As $D \rightarrow 0$, $sn \rightarrow tanh$, eventually, at T_M, $tanh$ splits into M_0 and

$-M_0$

$A > 0,\ D < 0$ As $D \to 0$ $cn \to M = 0$

$A < 0,\ D < 0$ As $D \to 0$, $cn \to sech$, eventually, at T_M, $sech$ splits into M_0 and $-M_0$

where it was postulated that $D = d(T - T_M)$.

Here, those results are extended to include some other geometries. The actual values of these solutions and their free energies were obtained numerically with the exception of the planar symmetry where both numerical and analytical calculations were performed. For the planar geometry we assumed the initial conditions to be given at the centre of the sample and equal to the maximum possible value of the solution and its derivative equal to 0. For other symmetries we assumed the maximal value at the boundary of the sample because the related ODE's have singularities at the origin. In our calculations we compared three different solutions having relatively simple symmetry properties for both D_x being positive and negative which are given below.

Geometry	$D_x > 0$	$D_x < 0$
Planar	$\xi = x_i$	$\xi = x$
Cylindrical	$\xi = (x_i + x_j)^{1/2}$	$\xi = (y^2 + z^2)^{1/2}$
Spherical	$\xi = (x^2 + y^2 + z^2)^{1/2}$	
Hyperbolical		$\xi = -x^2 + y^2 + z^2$

where $i, j = 1, 2, 3$. These surfaces of the constant value are illustrated in Fig. 3.1 for $D > 0$ and in Fig. 3.4 for $D < 0$.

To make this comparison easier a volume of the sample was fixed but the shape of the sample was slightly varied to make it more compatible with a given symmetry. The sample was assumed to be of cubic, spherical or cylindrical shape with the volume equal to 10^6 in lattice constant units. The model parameters were chosen as $b = -1.0$ $c = 100.0$ and $D_y = D_z = 1.0$. The value of the coefficient D_x was successively changed for comparison purposes and was taken to be $-100.0, -10.0, -1.0, 1.0$.

For the planar waves the free energy can be rewritten in the form:

$$\begin{aligned} F[M] &= \int_V d\vec{r}(\frac{b}{2}M^2 + \frac{c}{4}M^4 + \frac{D}{2}|\nabla M|^2) \\ &= A\int_L d\xi(\frac{b}{2}M^2 + \frac{c}{4}M^4 + \frac{D_\xi}{2}(\partial_\xi M)^2) \end{aligned} \tag{3.46}$$

where ξ is one of the equivalent directions in the sample. To simplify notation ξ is assumed to be equal to x in the following paragraphs. For the above form of the free energy functional, the planar wave solutions are taken from Winternitz et al [66]:

i) when $b < 0$ $D_x < 0$ and $K_1 < 0$

$$M(x) = \sqrt{-\frac{b}{c} + \sqrt{(\frac{b}{c})^2 - K_1}}\; cn(\sqrt{-\frac{c}{D_x}}((\frac{b}{c})^2 - K_1)^{1/4}(x - x_0), k)$$

where

$$k = \sqrt{\frac{-\frac{b}{c} + \sqrt{(\frac{b}{c})^2 - K_1}}{2\sqrt{(\frac{b}{c})^2 - K_1}}}$$

ii) when $b < 0$ $D_x < 0$ and $K_1 > 0$ or when $b < 0$ $D_x > 0$ and $K_1 > 0$

$$M(x) = \sqrt{-\frac{b}{c} + \sqrt{(\frac{b}{c})^2 - K_1}}\; dn(\sqrt{\frac{-c}{2D_x}}\sqrt{-\frac{b}{c} + \sqrt{(\frac{b}{c})^2 - K_1}}(x - x_0), k) \tag{3.47}$$

where

$$k = \sqrt{\frac{2\sqrt{(\frac{b}{c})^2 - K_1}}{-\frac{b}{c} + \sqrt{(\frac{b}{c})^2 - K_1}}}.$$

In both above equations K_1 is the integration constant. In the reduced ordinary differential equations the x coordinate does not appear explicitly. Thus, the second integration constant is implicitly given by fixing the position x_0 of one of the characteristic points of a solution which in this case is assumed to be the position of the maximum value of solutions. Using the above solutions, integrating by parts powers of the Jacobi transcendental functions and neglecting surface terms one obtains: when $b < 0$ $D_x < 0$ and $K_1 < 0$

$$F[M] = V[\frac{cK_1}{12} + \frac{b}{6}(\frac{b}{c} + \sqrt{(\frac{b}{c})^2 - K_1}) + \frac{2}{3}b\sqrt{(\frac{b}{c})^2 - K_1}\frac{E(k)}{K(k)}] \tag{3.48}$$

when $b < 0$ $D_x < 0$ and $K_1 > 0$ or $b < 0$ $D_x > 0$ and $K_1 > 0$

$$F[M] = V[\frac{cK_1}{12} + \frac{b}{3}(-\frac{b}{c} + \sqrt{(\frac{b}{c})^2 - K_1})\frac{E(k)}{K(k)}] \tag{3.49}$$

where $K(k)$ and $E(k)$ are the first and second complete Jacobi integrals and k is the modulus of the Jacobi functions. Because $k \to 1$ when $K_1 \to 0$ it is easy to see that the ratio $E(k)/K(k) \to 0$ and in the same limit the value of the free energy goes to 0. It is consistent with the assumed initial condition for the wave, namely that its maximum is fixed at the origin. Thus, this solution tends to the constant homogeneous solution. The results published by Winternitz et al [66] were obtained under the assumption that the fixed value at the origin is equal to the average of the maximum and minimum values. In such a case the solution tends to the "kink" solution and the total free energy has a positive limit.

The first and quite surprising result is that the value of the free energy functional does not explicitly depend on the magnitude of the coefficient D_x. This fact holds true as long as one does not consider the cutoff condition. When the cutoff condition is imposed on planar wave types one gets the band of solutions that becomes very narrow when D_x tends to 0.

One should always remember, however, that there are singular solutions of the reduced ordinary differential equation. The family of these solutions is related to spatially homogeneous distributions of the order parameter. The value of the free energy functional corresponding to these solutions is equal to

$$\begin{aligned} F[M] &= V[\frac{b}{2}M^2 + \frac{c}{4}M^4] \\ &= V\frac{3b^2}{4c} \quad or \quad -V\frac{b^2}{4c} \end{aligned}$$

We are dealing with the case where $c > 0$, therefore, it is always the second value that has a lower energy.

The neglect of the surface terms leads to an error which should be smaller than twice the free energy of the wave over the length of one period. It should be then extremely good when one deals with 10^6 periods in the sample.

The results of our calculations are presented in Figs. 3.5 - 3.7 in which the free energy is plotted against the integration constant. In the case of planar symmetry the integration constant is directly related to the amplitude of the solution. This relation is not true for other geometries. In Fig. 3.5 we can see that the free energy of planar waves is weakly dependent on the value of the coefficient $D_x < 0$. The ripples appearing on the plot are related to the coincidence of the length of the sample and the multiplicity of the wavelength of the solution. These ripples disappear when we use an approximation where averages are calculated. The same comments are applicable to the free energy of planar waves when $D_x > 0$ Fig. 3.6. The value of the constant of integration is bounded from above by the smaller of the two values: the cutoff value and the value $(\frac{b}{c})^2 = 0.001$. In the case $D_x < 0$ the lower limit is given by the lower cutoff value while in the case $D_x > 0$ it is 0. We can see that the free energy has its maximum at 0, is monotonically increasing for the integration constant < 0 and monotonically decreasing otherwise. The value of the free energy for $D_x < 0$ is less than -4.5×10^3 in Fig. 3.5 and is of the order -1.1×10^3 for $D_x > 0$. The values of the free energy for solutions with cylindrical symmetry for the comparable values of the initial conditions were in the range $(-100, 0)$ while those for solutions with spherical symmetry were slightly above 0. Values of the

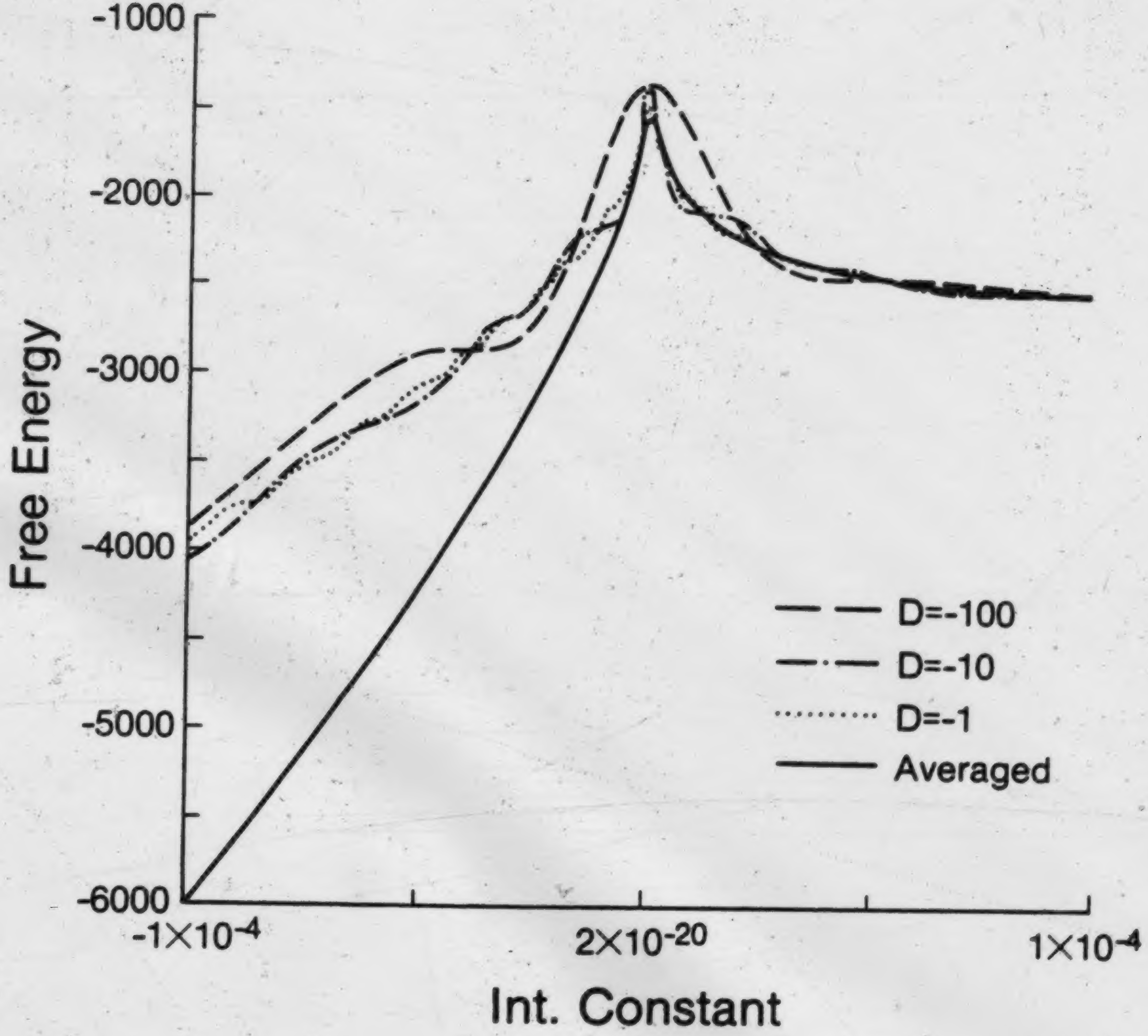

Figure 3.5: Numerical values of the free energy for plane waves, propagating in the x direction, solutions to the Eq. $D\partial_{xx}M = 100.M^2 - M^4$ for different values of D.

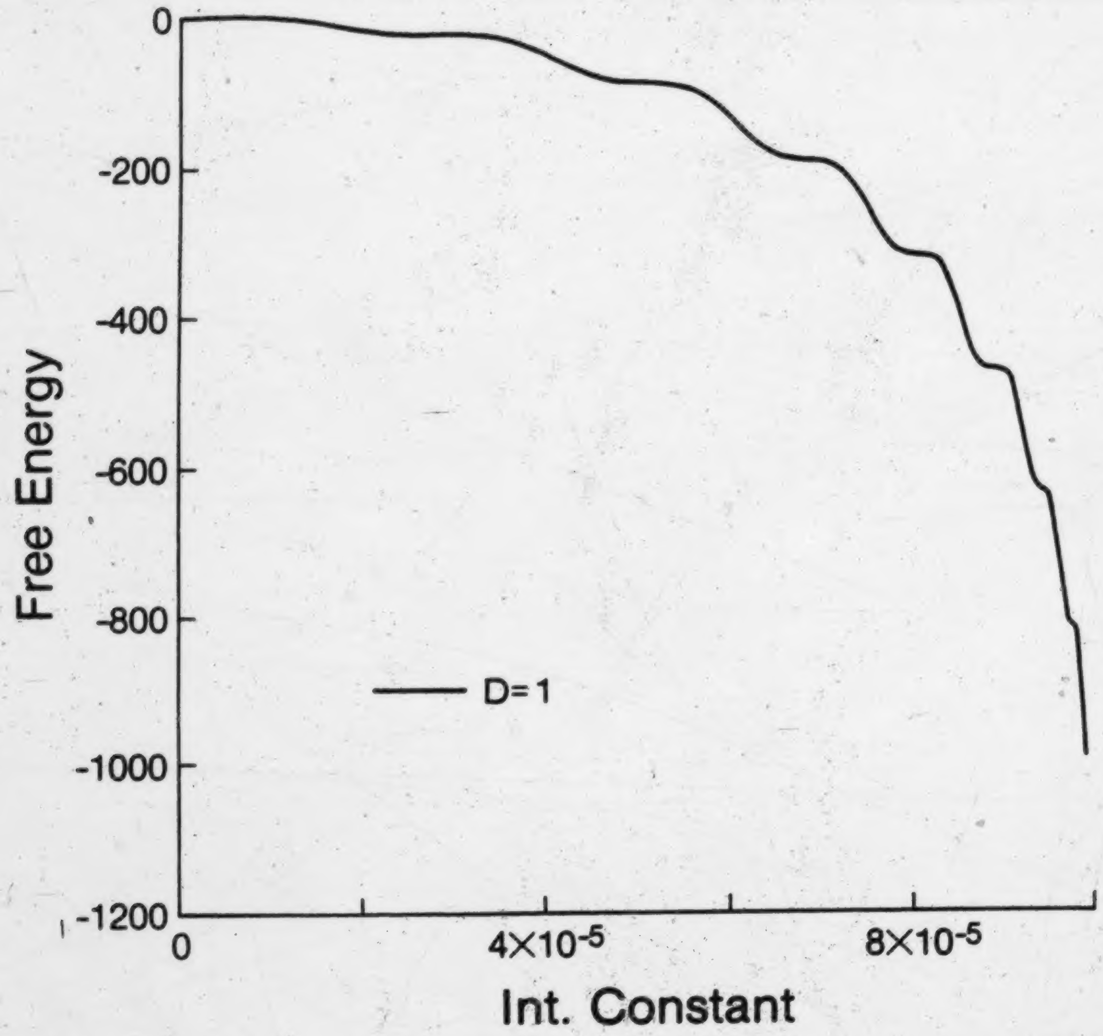

Figure 3.6: Numerical values of the free energy for plane waves, propagating in the x direction, solutions to the Eq. $\partial_{xx}M = 100.M^2 - M^4$.

free energy functional for the solutions with hyperbolic symmetry are of the order of $-1. \times 10^8$ as can be seen in Fig. 3.7.

We can see that the lowest free energy corresponds to solutions with hyperbolic symmetry followed by those with planar symmetry (first with $D_x < 0$ then with $D_x > 0$) and finally by those with cylindrical and spherical symmetries.

The above results seem to be quite reasonable with the exception of the unexpectedly low value of the free energy for the hyperbolic type solutions. As far as we know this type of solution is not commonly observed. We think that such low values are related to the fact that surfaces of constant magnetization are getting closer when they approach the singular conical surface $-x^2 + y^2 + z^2 = 0$. If the magnetization is not constant outside and/or inside of this surface the gradient term in the free energy functional may become very large in the positive or negative direction because of an increased density of surfaces. On the other hand, we assumed that the solution should not significantly vary on a scale smaller then the lattice constant. It seems then reasonable to assume that the solution with such symmetry should be close to the spatially homogeneous outside and/or inside the above cone.The free energy is equal then to that of the mean field solution -2.5×10^3.

3.4 Experimental applications

There are some experimental results related to the above discussed theory. Most of the experiments investigating the magnetization of solids are based on some forms of scattering. The scattering experiments are described in terms of planar waves and consequently only planar geometry can be investigated by those methods. Therefore, in most of the experimental results all the attention is concentrated on the planar wave solutions.

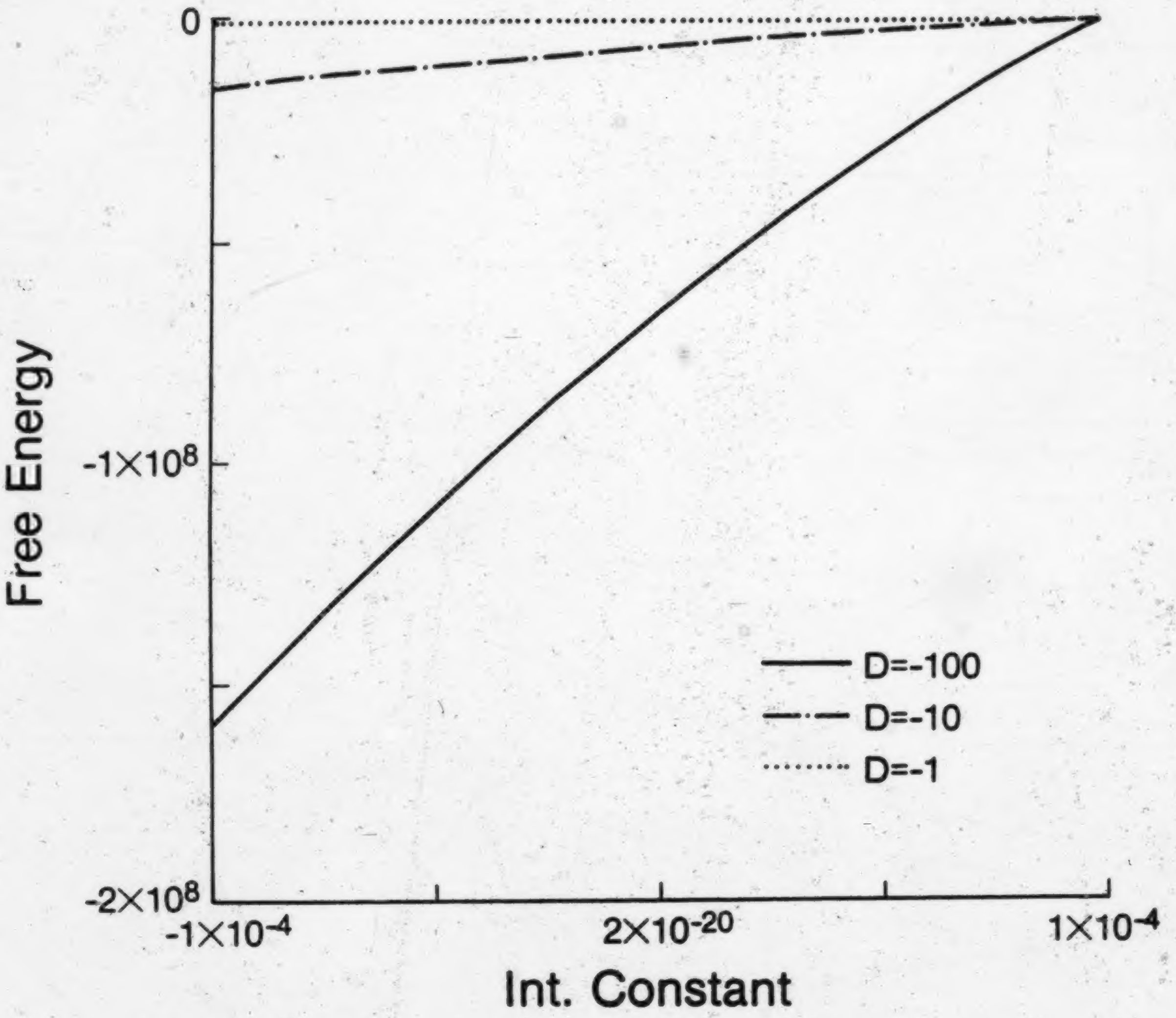

Figure 3.7: Numerical values of the free energy for hyperbolic waves, propagating in the x direction, solutions to the Eq. $D\partial_{xx}M + \partial_{yy}M + \partial_{zz}M = 100.M^2 - M^4$ for different values of D.

The experiment which can be described by Landau-Ginzburg theory with positive signature (case $D_x > 0$) was performed by D. LaGraffe et al [35] and later theoretically discussed by P.A. Dowben et al [14]. In this experiment a thin overlayer of Tb is deposited on the substrate of Ni and on Cu. The whole system is investigated in the range of temperatures around Curie temperature for Tb. The anisotropy of the 5p levels was then measured using angle resolved photoemission without spin detection of photoelectrons. It was discovered that there is a significant difference in this anisotropy as a function of the thickness of the overlayer in the case of different substrates.

This phenomenon can be explained by considering magnetic interactions between the substrate and the overlayer. For Tb at temperature above its Curie point but below Curie temperature for Ni the ferromagnetic order in Ni induces some magnetic orientation in the Tb overlayer. The measured asymmetry vanishes exponentially. It can be shown that this behaviour can be described using a one-dimensional order parameter.

Using boundary conditions imposed on the logarithmic derivative of magnetization P.A. Dowben et al [14] were comparing these results to the exponential curve. They obtained the correlation length $2.5 \pm 0.7\mathring{A}$.

In this thesis the main equation describing magnetization [14] work was obtained for the planar wave solutions as Eq. (3.4). This equation does not have exponential solutions but it has the bump type solution as a solution to Eq. (3.4). This solution is inversely proportional to hyperbolic cosine and therefore the dominant behaviour of the bump solutions is exponential.

Some more detailed theoretical results in the case $D_x > 0$ describing the characteristic quantities of the planar-wave type solutions of the Landau-Ginzburg equation were given by P. Winternitz et al [68]. The field dependence of the mean magnetization, of the wavelength and of the free energy for many type elliptic solutions are presented in this work. These quantities can be measured experimentally.

The case $D_x < 0$ was discussed by W.I. Khan [31]. He attempted in his calculations to explain the experimental data obtained by J.M. Barandiaran et al [7]. J.M. Barandiaran et al discovered that below $T_N = 18K$ the magnetic phase of $PrNi_2Si_2$ rare earth alloys is sinusoidally modulated. In this temperature a crystal of this alloy reveals a large uniaxial magnetocrystlline anisotropy. Ni is non-magnetic and Pr moments are along the c-axis with a maximum value of $2.6 \pm 0.1 \mu_B$ at $5.5K$.

W.I. Khan discussed the possibility of describing a sinusoidal magnetic phase in rare earth alloys using Landau-Ginzburg equations. He has used the free energy functional in the form

$$F = \frac{1}{2}AM^2 + \frac{1}{4}BM^4 + \frac{1}{6}CM^6 + G(\nabla M)^2.$$

The coefficients of this functional were rescaled to the form $\beta = A/2D$, $\gamma = B/4D$, $\delta = C/6D$, and $D = -G$.

He has also shown that it is possible to find a solution which closely follows the experimental curve of the nonhomogeneous magnetization obtained in [7]. He managed to fit both the period and magnetization to experimental data. However, questions of numerical values of its coefficients and their justification remain still open.

The equation used by W.I. Khan in [31] to describe magnetization was also obtained in this thesis and is listed in Table 8 as Eq. 2. Its solutions were discussed together with solutions of Eq. (3.4). It has many solutions oscillating around zero as well as some bump type solutions.

To observe more complicated geometries one has to use different experimental methods. These would include the direct observations of the domain shapes by

Bitter's and Faraday's methods. However, it is difficult to say whether the one-dimensional order parameter is sufficient to describe such geometries.

CHAPTER FOUR

CONCLUSIONS

In previous chapters results of the symmetry reduction applied to the time- and space-dependent Landau-Ginzburg equation with two possible signatures were presented. These results contain a classification of all possible subalgebras that lead to reductions of these equations to equations of a lower order or to equations in fewer variables.

All reductions to algebraic and to ordinary differential equations were explicitly performed. Solutions of ordinary differential equations of second order that satisfied the Painlevé Test were given. Also methods for integration of the first order differential equations were also given. Our results have suggested that Landau theory of phase transitions contains a large number of continuous symmetries which have been investigated for the first time.

The limitation of considerations to the discrete subgroups of the group of translations can be possibly justified on a microscopic scale. The proper description then should be completely microscopic and one should start from a lattice model.

On the other hand, there is no apparent reason to limit the analysis to this relatively narrow class of formations when one discusses the coarse-grained approximations on the mesoscopic scale. The symmetry method provides an appropriate method of investigation of these cases.

However, in a similar way to the case of topological classification [34], symmetry analysis does not provide one with the ability to predict which of the obtained symmetries is actually realised. This question should be solved by comparing energies of solutions with different geometries. The results of a preliminary attempt by the author at numerical evaluation of these energies associated with 3.3 seem to suggest that a significant role is played by regions close to singularities and boundaries. For orbits with negative curvature one can see that because surfaces of constant value are getting arbitrarily close in the physical space the free energy may become divergent.

There are many remaining problems for which solutions were not even attempted in this thesis. For example, discussion of stability of solutions, the hierarchy of phase transitions, or statistical properties around each particular solution.

I would like to mention some directions of possible extension of this thesis. Probably the most interesting one, at least in my opinion, is the extension of this investigation to include more dimensional (e.g. complex) order parameter. The case investigated here is possibly too simple in the sense that despite its value as a model systems one would not expect to find any practical applications of the one-dimensional non-conserved real order parameter. Therefore, the actual value of the above reduction should be more measured in terms of the idealized model than actual representation of experimental results. Another problem is related to the lack of precise mathematical understanding of continuous approximations which necessarily appear in a coarse-grained procedure. Because of this it is not clear what are the limits of applicability of the obtained results. In many places [23,43,40] it is suggested that this type of approximation should make sense only for slowly varying quantities. However, at the same time many authors utilize the same method to describe results on a length scale comparable with average atomic distances [31,45]. It is quite possible that the dynamical behaviour can be modelled even on very short scales when there is very weak time dependence (a situation which is dynamic locally can be averaged using the continuous approximation while the global character of solutions remains almost the same). These problems seem to be quite difficult and as far as I know open to further investigations.

The results presented in this thesis proved that the symmetry reduction method is a valuable tool in the analysis of physical problems. It gives one means not only to look for special solutions but also to predict the possible hierarchy of symmetries appearing in the sequence of patterns.

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APPENDIX A

BASIC DEFINITIONS AND THEOREMS

1.1 Basic definitions

Definition A.1 *An* **r-parameter local Lie group** *consists of connected open subsets* $V_0 \subset V \subset R^r$ *containing the origin 0, and smooth maps* $m : V \times V \to R$, *defining the group operation, and* $i : V_0 \to V$, *defining the group inversion, with the following properties.*

1. **Associativity**. *If* $x, y, z \in V$, *and also* $m(x,y)$ *and* $m(y,z)$ *are in* V, *then* $m(x, m(y,z)) = m(m(x,y), z)$.

2. **Identity Element**. *For all* x *in* V, $m(x,0) = x = m(0,x)$.

3. **Inverses**. *For each* x *in* V , $m(x, i(x)) = 0 = m(i(x), x)$.

Definition A.2 *Let* M *be a smooth manifold. A* **local group of transformations** *acting on* M *is given by a local Lie group* G, *an open subset* U, *with*

$$\{e\} \times M \subset U \subset G \times M$$

which is the domain of definition of the group action, and a smooth map $\Psi : U \to M$ *with the following properties:*

1. *If* $(h,x) \in U$, $(g, \Psi(h,x)) \in U$, *and also* $(g \cdot h, x) \in U$, *then* $\Psi(g, \Psi(h,x)) = \Psi(g \cdot h, x)$

2. *For all* $x \in M$, $\Psi(\epsilon, x) = x$

3. *If* $(g,x) \in U$, *then* $(g^{-1}, \Psi(g,x)) \in U$ *and* $\Psi(g^{-1}, (g,x)) = x$

Definition A.3 *Let G be a local group of transformations acting on a manifold M. A subset $S \subset M$ is called* **G-invariant**, *and G is called a* **symmetry group of** S, *if whenever $x \in S$, and $g \in G$ is such that $g \cdot x$ is defined, then $g \cdot x \in S$.*

Definition A.4 *Let G be a local group of transformations acting on a manifold M. A function $F : M \to N$, where N is another manifold, is called a* **G-invariant function** *if for all $x \in M$ and all $g \in G$ such that $g \cdot x$ is defined, $F(g \cdot x) = F(x)$.*

Definition A.5 *Let G be a local group of transformations acting on the manifold M. A subset $S \subset M$ is called* **locally G-invariant** *if for every $x \in S$ there is a neighbourhood $\tilde{G}_x \subset G_x$ of the identity in G such that $g \cdot x \in S$ for all $g \in \tilde{G}_x$. A smooth function $F : U \to N$, where U is some open subset of M, is called* **locally G-invariant** *if for each $x \in U$ there is a neigbourhood $\tilde{G}_x \subset G_x$ of e in G such that $F(g \cdot x) = F(x)$ for all $g \in \tilde{G}_x$. F is called* **globally G-invariant** *if $F(g \cdot x) = F(x)$ for all $x \in U$, $g \in G$ such that $g \cdot x \in U$.*

Definition A.6 *Let S be a system of differential equations. A* **symmetry group** *of the system S is a local group of transformations G acting on the open subset M of the space of independent and dependent variables for the system with the property that whenever $u = f(x)$ is a solution of S, and $g \cdot f$ is defined for $g \in G$, then $u = g \cdot f(x)$ is also a solution of the system. (By* **solution** *we mean any smooth solution $u = f(x)$ defined on any subdomain $\Omega \subset X$).*

Definition A.7 *Let $M \subset X \times U$ be an open subset and suppose v is a vector field on M, with corresponding (local) one-parameter group* $\exp(\epsilon v)$. *The* **n-th prolongation** *of v, denoted by $pr^{(n)}v$, will be a vector field on the $n-jet$ space $M^{(n)}$, and is defined to be the infinitesimal generator of the corresponding prolonged one-parameter group $pr^{(n)}[\exp(\epsilon v)]$. In other words,*

$$\forall (x, u^{(n)}) \in M^{(n)},\ pr^{(n)}v \mid_{(x,u^{(n)})} = \frac{d}{d\epsilon} \Big|_{\epsilon=0} pr^{(n)}[\exp(\epsilon v)](x, u^{(n)})$$

Definition A.8 *A system of n-th order differential equations* $\Delta(x, u^{(n)}) = 0$ *is* **locally solvable at the point**

$$(x_0, u_0^{(n)}) \in S_\Delta = \{(x, u^{(n)} : \Delta(x, u^{(n)}) = 0\}$$

if there exists a smooth solution $u = f(x)$ *of the system, defined for* x *in a neighbourhood of* x_0*, which has the prescribed "initial conditions"* $u_0^{(n)} = pr^{(n)} f(x_0)$*. The system is* **locally solvable** *if it is locally solvable at every point of* S_Δ*. A system of differential equations is* **nondegenerate** *if at every point* $(x_0, u_0^{(n)}) \in S_\Delta$ *it is both locally solvable and of maximal rank.*

1.2 Basic theorems

Theorem A.9 *If* G *acts on* M*, and* $F : M \to R^l$ *is a smooth function, then* F *is a* G*-invariant function if and only if every level set* $F(x) = c$, $c \in R^l$*, is a* G*-invariant subset of* M*.*

Theorem A.10 *Let* G *be a connected local Lie group of transformations acting on the* m*-dimensional manifold* M*. Let* $F : M \to R^l$, $l \leq m$*, define a system of algebraic equations* $F_\nu = 0, \nu = 1, \ldots, l$ *and assume that the system is of maximal rank, meaning that the Jacobian matrix* $\frac{\partial F_\nu}{\partial x^k}$ *is of rank* l *at every solution* x *of the system. Then* G *is a symmetry group of the system if and only if* $v[F_\nu(x)] = 0$, $\nu = 1, \ldots, l$ *whenever* $F(x) = 0$*, for every infinitesimal generator* v *of* G

Theorem A.11 *Let* M *be an open subset of* $X \times U$ *and suppose* $\Delta(x, u^{(n)}) = 0$ *is an* n*-th order system of the differential equations defined over* M*,with corresponding subvariety,* $S_\Delta = \{(x, u^{(n)}) | \Delta(x, u^{(n)}) = 0\}$, $S_\Delta \subset M^{(n)}$*. Suppose* G *is a local group of transformations acting on the* M *whose prolongation leaves* S_Δ *invariant, meaning that whenever* $(x, u^{(n)}) \in S_\Delta$*, we have* $pr^{(n)} g \cdot (x, u^{(n)}) \in S_\Delta$ *for all* $g \in G$

such that this is defined. Then G is a symmetry group of the system of differential equations in the sense of Definition A.6.

Theorem A.12 *Suppose*

$$\Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, \ldots, l,$$

is a system of differential equations of maximal rank over $M \subset X \times U$. If G is a local group of transformations acting on M, and

$$pr^{(n)}v[\Delta_\nu(x, u^{(n)}] = 0, \qquad \nu = 1, \ldots, l, \qquad whenever \qquad \Delta(x, u^{(n)}) = 0,$$

for every infinitesimal generator v of G, then G is a symmetry group of the system.

Theorem A.13 *Let $\Delta_\nu(x, u^{(n)}) = 0$, be a nondegenerate system of differential equations. A connected local group of transformations G acting on an open subset $M \subset X \times U$ is a symmetry group if and only if*

$$pr^{(n)}v[\Delta_\nu(x, u^{(n)})] = 0, \qquad \nu = 1, \ldots, l \qquad whenever \qquad \Delta(x, u^{(n)}) = 0,$$

for every infinitesimal generator v of G.

Theorem A.14 *Let G act semi-regularly on the m-dimensional manifold M with s-dimensional orbits. If $x_0 \in M$, then there exist precisely $m - s$ functionally independent invariants $\xi^1(x), \ldots, \xi^{m-s}(x)$ defined in a neighbourhood of x_0. Moreover, any other invariant of the group action defined near x_0 is of the form*

$$\xi(x) = F(\xi^1(x), \ldots, \xi^{m-s}(x))$$

for some function F. If the action of G is regular, then the invariants can be taken to be globally invariant in a neighbourhood of x_0.

Theorem A.15 *Let G be the symmetry group of a system of differential equations Δ and let $H \subset G$ be an s-parameter subgroup. If $u = f(x)$ is an H-invariant solution to Δ and $g \in G$ is any other group element, then the transformed function $u = \tilde{f}(x) = g \cdot f(x)$ is an $\tilde{H}$-invariant solution, where $\tilde{H} = gHg^{-1}$ is the conjugate subgroup to H under g.*

Theorem A.16 *Let H and $\tilde{H}$ be connected, s-dimensional subgroups of the Lie group G. Then $\tilde{H} = gHg^{-1}$ are conjugate subgroups if and only if $\tilde{h} = Ad\, g(h)$ are conjugate algebras.*